Polycentric Geometry: From the Universal Proportion of a Point and a Sphere to Networks of Intertwined Spheres

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Abstract

In the present work, we introduce and rigorously prove the notion of the *universal proportion* for a point inside a sphere — a dimensionless ratio connecting the point, the sphere, and their images under inversion. It is shown that this ratio not only completely determines the geometry of the "point–sphere" configuration but also naturally generalises to the case of *polycentric systems* — finite sets of intertwined spheres with a common interior point. We develop the formalism of polycentric coding, groups of inversion symmetries, and harmonic networks. Connections with conformal and projective geometry are established. The results have applications in reciprocity physics, computer graphics, and the theory of fractal packings.

1 Introduction

Classical sphere geometry treats the sphere as a single object: the boundary of a ball, the set of points at a fixed distance from a centre. Inversion with respect to a sphere, polar correspondence, and harmonic division of chords are usually presented as separate facts [1, 2]. The present work originates from the intuitive observation: *any point inside a sphere is computable by the same laws as the sphere itself*. This observation is formalised as the **Theorem of Universal Proportion**, which constitutes the first part of the paper. The second part generalises this principle to the case of many spheres inscribed within each other and intersecting — giving rise to **polycentric geometry**, in which a point is encoded by a set of proportions relative to all spheres of the system. The concluding part unifies both constructions into a single formalism.

Inversion is a classical conformal transformation with deep connections to potential theory and harmonic functions [1, 4]. It preserves the family of spheres and hyperplanes in $\mathbb{R}^n$ and has been extensively used in the method of images for solving boundary value problems [1, 4]. The harmonic division property of inversion has been studied in the context of Apollonian circles and pencils [3]. More recently, inversive geometry has become a fundamental tool in the construction of self-similar space-filling packings, where the inversion algorithm generates Apollonian gaskets and their generalisations [5, 6]. These packings are intimately connected to Kleinian groups and their limit sets

[7, 8]. The present work builds on these classical and modern developments to introduce a unified framework — polycentric geometry — that bridges the arithmetic of points, the geometry of spheres, and the topology of inversion-generated networks.

2 Part I. The Theorem of Universal Proportion (Point and Sphere)

2.1 Definitions

Let $\mathbb{R}^3$ be three-dimensional Euclidean space with a fixed origin O and the standard Euclidean norm $|\cdot|$. A **sphere** S of radius $R > 0$ with centre O is the set

S := { X $\in \mathbb{R}^3$ | $|OX| = R\}$.

Inversion Inv_S with respect to S is the map $\mathrm{Inv}_S : \mathbb{R}^3 \setminus \{O\} \to \mathbb{R}^3 \setminus \{O\}$ which sends a point $P \neq O$ to the point $P' = \mathrm{Inv}_S(P)$ lying on the ray OP such that

|OP| $\cdot |OP'| = R^2$.

This is the classical inversion first employed by Lord Kelvin in potential theory [1]. It is a conformal transformation that preserves the family of spheres and hyperplanes [3, 2].

The **universal proportion** $\lambda(P)$ for a point $P \neq O$ with respect to S is the dimensionless quantity

$\lambda(P) := \frac{|OP|}{R}$.

The **polar plane** of $P \neq O$ with respect to S is the plane π_P given by

$\cdot \vec{OX} = R^2$.

Geometrically, it is perpendicular to $\vec{OP}$ and is at distance $R/\lambda(P)$ from O.

The **cross ratio** (anharmonic ratio) of four collinear points A, B, C, D is denoted $(A, B; C, D)$ and defined as

$$\text{(A,B;C,D)} = \text{AC}_{\overline{CB:\frac{AD}{DB}},}$$

where segments are taken with sign. The harmonic relation corresponds to the value -1. Cross ratios are invariant under Möbius transformations, including inversions [3].

2.2 Statement of the Theorem

Theorem 2.1 (Universal Proportion)**.** *Let S be a sphere of radius $R > 0$ with centre O, and let P satisfy $0 < |OP| < R$ (an interior point, distinct from the centre). Then the following hold.*

(1) Inversion symmetry. *The image $P' = \mathrm{Inv}_S(P)$ lies outside S and satisfies*

$$\lambda(P) \cdot \lambda(P') = 1, \qquad \textit{or equivalently} \qquad \tfrac{|OP|}{R} = \tfrac{R}{|OP'|}.$$

The map Inv_S is involutive: $\mathrm{Inv}_S(P') = P$.

(2) Polar reciprocity. *The polar plane π_P does not intersect S, and the distance from the centre to it is $R/\lambda(P)$. The point P is uniquely recoverable from π_P.*

(3) Harmonic structure. *For any line through P intersecting S in two distinct points A, B, we have*

(A,B; P, P’) = -1,

where $P' = \mathrm{Inv}_S(P)$.

(4) Concentric foliation. *The point P determines a sphere S_r of radius $r = |OP|$ with the same centre O; S_r is the image of S under the homothety $X \mapsto (r/R)X$. The equation of S_r has the same form as that of S, differing only in the radius value.*

This theorem establishes that the geometry of a point inside a sphere is governed by the same algebraic structure as the sphere itself. The universal proportion $\lambda(P)$ is the invariant that encodes this duality.

2.3 Proof of the Harmonic Structure (Point 3)

Consider a line ℓ through P intersecting S in two distinct points A, B. Extend ℓ to the projective line $\mathbb{P}^1$. Inversion with respect to S induces on ℓ an involutive projective transformation ι, since points of S are fixed by inversion: for $X \in S$, $|OX| = R$, hence $|OX'| = R^2/R = R$ and X' lies on the same ray, so $X' = X$. Thus A and B are fixed points of ι. The unique involution of $\mathbb{P}^1$ with two distinct fixed points is a harmonic homology, for which any pair $(X, \iota(X))$ harmonically separates the fixed points. Applying this to $X = P$ and $\iota(P) = P'$, we obtain $(A, B; P, P') = -1$. This property is classical in inversive geometry and is closely related to the construction of Apollonian circles [3].

2.4 Generalisation to $\mathbb{R}^n$

All definitions and assertions of Theorem 2.1 extend to $\mathbb{R}^n$ $(n \geq 2)$ without change, replacing:

- "sphere" by an $(n-1)$-dimensional sphere,
- "plane" by a hyperplane of dimension $n-1$.

The equations and proofs remain valid verbatim. This is consistent with the conformal compactification of $\mathbb{R}^n$ and the invariance of harmonicity under inversion with appropriate conformal weights [1, 4].

3 Part II. Polycentric Geometry (Intertwined Spheres)

3.1 Motivation and Definitions

The idea of the universal proportion naturally leads to a generalisation: if a point is "computable" with respect to one sphere, how is it described with respect to *several* spheres simultaneously? Consider a system of spheres that are nested, tangent, or intersecting, but share a common interior region. Each sphere defines its own proportion, and together they form a **polycentric code** of the point.

Definition 3.1 (Polycentric system)**.** *A **polycentric system** $\mathcal{S}$ is a finite set of spheres $\mathcal{S} = \{S_1, \ldots, S_m\}$ $(m \geq 1)$ in $\mathbb{R}^n$ with centres O_i and radii $R_i > 0$, such that:*

1. *$\mathcal{I} := \bigcap_{i=1}^m \{X \in \mathbb{R}^n \mid |O_iX| < R_i\} \neq \varnothing$ (there exists a common interior point);*
2. *All centres are distinct: $O_i \neq O_j$ for $i \neq j$.*

The condition of distinct centres prevents degeneracy. We do not require that no sphere is contained within another; such redundancy is technical and does not affect the essential structure of the polycentric code. This definition is inspired by the use of inversion spheres in the construction of Apollonian packings and other space-filling fractals [5, 6].

3.2 Polycentric Code

For a point $P \in \mathcal{I}$, define its **polycentric code** $\Lambda(P)$ as the vector of universal proportions with respect to each sphere of the system:

$$\Lambda(P) := (\lambda_1(P), \ldots, \lambda_m(P)), \qquad \lambda_i(P) = \tfrac{|O_iP|}{R_i}.$$

Since P lies inside all spheres, $\lambda_i(P) \in (0,1)$ for all i.

3.3 Uniqueness of the Polycentric Code

Theorem 3.2 (Uniqueness of the polycentric code)**.** *Let $\mathcal{S}$ be a polycentric system in $\mathbb{R}^n$. If the centres $\{O_i\}$ do not lie on a common $(n-2)$-dimensional sphere and $m \geq n+1$, then the map $\Lambda : \mathcal{I} \to (0,1)^m$, $P \mapsto \Lambda(P)$, is injective.*

Proof. The equality $\Lambda(P) = \Lambda(Q)$ implies $|O_iP|/R_i = |O_iQ|/R_i$ for all i, hence $|O_iP| = |O_iQ|$. Thus P and Q are equidistant from all centres O_i. In $\mathbb{R}^n$, the set of points equidistant from $n+1$ points in general position consists of at most one point (the intersection of $n+1$ hyperspheres with centres O_i, having at most two points, but the condition of lying inside the fixed domain $\mathcal{I}$ selects one). The condition that the centres do not lie on a common $(n-2)$-dimensional sphere guarantees that the system of equations $|O_iP| = \text{const}$ is non-degenerate. Hence $P = Q$. □

□

This result is analogous to trilateration in navigation and is central to the practical applicability of polycentric codes.

3.4 The Group of Polycentric Symmetries

Theorem 3.3 (Group of inversion symmetries)**.** *Let* Inv_i *be the inversion with respect to* $S_i \in \mathcal{S}$*. Then the set of all finite compositions* $\mathrm{Inv}_{i_1} \circ \cdots \circ \mathrm{Inv}_{i_k}$ *forms a subgroup* $G_{\mathcal{S}}$ *of the conformal group of* $\mathbb{R}^n$*. This group acts on the extended space* $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ *and generates the orbit of the initial sphere system* $\mathcal{S}$ *under repeated inversions, preserving the conformal structure.*

Proof. Each inversion is a conformal transformation; the composition of conformal transformations is conformal. Closure under composition holds by construction. The identity element is the identity transformation (empty composition). The inverse of $\mathrm{Inv}_{i_1} \circ \cdots \circ \mathrm{Inv}_{i_k}$ is $\mathrm{Inv}_{i_k} \circ \cdots \circ \mathrm{Inv}_{i_1}$, since each inversion is involutive. The action on $\widehat{\mathbb{R}}^n$ is well-defined because inversions are extended to the point at infinity. □

□

This group is a discrete subgroup of the Möbius group. In the case of Apollonian packings, such groups are Kleinian groups whose limit sets are the Apollonian gaskets [7, 8]. The group $G_{\mathcal{S}}$ generated by inversions in the dual circles of a circle packing is precisely the Apollonian group studied in [7].

3.5 The Harmonic Network

Theorem 3.4 (Harmonic network)**.** *For any sphere S_i and any line ℓ through P intersecting S_i in two points A_i, B_i, we have*

$$(A_i, B_i;\, P, P_i') = -1,$$

where $P_i' = \mathrm{Inv}_i(P)$*. The collection of all such harmonic quadruples for all spheres of the system and all lines through P forms a harmonic network invariant under the action of $G_{\mathcal{S}}$.*

Proof. The first assertion follows directly from Theorem 2.1(3) applied to each sphere S_i. For invariance: any element $g \in G_{\mathcal{S}}$ is a conformal transformation. Conformal transformations preserve the cross ratio of four points; hence a harmonic quadruple is mapped to a harmonic quadruple. The image of a line is a generalised circle; the harmonic structure of the network is preserved. □

□

This harmonic network generalises the classical notion of Apollonian circles and pencils [3]. It is a discrete conformal structure generated by the polycentric system.

3.6 Recovering the System from an Interior Point

Theorem 3.5 (Recovering the system). *Suppose the radii $\{R_i\}$ of all spheres of the system $\mathcal{S}$ are given. Then from the polycentric code $\Lambda(P)$ the point P and the centres $\{O_i\}$ are recoverable up to rigid motion of the space. If the radii are unknown but the code $\Lambda(P)$ is known, the system $\mathcal{S}$ is recoverable up to similarity.*

Proof. The code $\Lambda(P)$ gives the values $|O_iP| = \lambda_i R_i$. With known $\{R_i\}$, this is a set of distances from P to each centre O_i. The problem of recovering P and $\{O_i\}$ from mutual distances is the trilateration problem. For $m \geq n+1$ and centres in general position, the solution is unique up to rigid motion. If the radii are unknown, the proportions λ_i determine the ratios $|O_iP|/R_i$; multiplying all lengths by a common factor $t > 0$ does not change the code. Hence the system is determined up to similarity. □

□

This theorem is the geometric analogue of the reconstruction of a point configuration from distance data, with the universal proportions playing the role of normalised distances.

4 Part III. Unified Synthesis

4.1 The Point as a Collapsed System of Spheres

The Theorem of Universal Proportion (Part I) showed that a single point inside a single sphere is indistinguishable from the sphere in terms of algebraic structure: the equation of the sphere S_r of radius $r = |OP|$ has the same form as the equation of S_R. Polycentric geometry (Part II) demonstrates that a point seen through a system of intertwined spheres carries the codes of all spheres simultaneously. The union yields a fundamental principle:

A point in the intersection of spheres is equivalent to the entire system of spheres up to similarity and conformal symmetry.

This principle is deeply connected to the theory of Kleinian reflection groups and Apollonian packings, where the limit set of a group generated by sphere inversions encodes the fractal structure of the packing [7, 5, 6]. In this framework, a point of the limit set carries information about the entire packing through the group action.

4.2 Conclusion

In this work, we have built a bridge between classical inversive geometry and a new domain — polycentric geometry. We introduced the notion of universal proportion, rigorously proved its properties for a single sphere, and generalised them to arbitrary systems of intertwined spheres. We obtained results on the uniqueness of polycentric coding, the structure of the conformal symmetry group, and harmonic networks.

These constructions are applicable in physics (the method of images, where Kelvin inversion transforms boundary value problems [1, 4]), computer graphics (spherical parametrisations), and fractal theory (Apollonian packings, where the inversion algorithm generates self-similar space-filling structures [5, 6], and limit sets of Kleinian groups [7, 8]). The polycentric framework provides a unified language for studying configurations of spheres, their inversion-generated symmetries, and the harmonic structures that arise from them.

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