Prime-Sum-Parametrized Configurations and Conditional Goldbach Representations with Fixed Differences

Tishkov Vladislav
tv8805730@gmail.com

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Abstract

We introduce a novel family of k-tuples $\Phi_S(m) = (m\Sigma_S - p)_{p\in S}$,
parametrized by a finite set S of distinct primes whose sum Σ_S is also prime.

We provide a complete local admissibility analysis for the associated linear forms.

We prove that *if* an admissible such set S exists with $|S| \geq C(\nu)$
(where $C(\nu)$ is the effective constant from the Maynard–Tao theorem), *then*
there exist infinitely many integers m for which at least ν components of
$\Phi_S(m)$ are simultaneously prime. Consequently, every such S conditionally
yields infinitely many even integers possessing at least ν distinct Goldbach
representations, with the differences between the corresponding prime components
fixed and determined by S. We provide an explicit unconditional example for
$k = 3$ and discuss the status of the existence problem for larger k.

1 Introduction

1.1 Background

Let $\mathbb{P}$ denote the set of prime numbers. For an even integer $N \geq 4$, a *Goldbach representation* is a pair $(p, q) \in \mathbb{P} \times \mathbb{P}$ such that $N = p+q$, where we identify (p, q) with (q, p) when $p \neq q$. The Goldbach conjecture — that every even $N \geq 4$ has at least one such representation — remains unproven.

A landmark result due to Maynard [4] (building on the breakthrough of Zhang [6] and the earlier work of Goldston, Pintz, and Yıldırım [2]) established that for any fixed $\nu \geq 2$, every admissible k-tuple of linear forms with k sufficiently large attains at least ν simultaneous prime values infinitely often.

Theorem 1.1 (Maynard [4], Theorem 3.1)**.** *Let $\mathcal{L} = \{L_1, \ldots, L_k\}$ be an admissible k-tuple of linear forms $L_i(n) = a_i n + b_i$ with integer coefficients satisfying $a_i > 0$ and $\gcd(a_i, b_i) = 1$ for all i. For any fixed integer $\nu \geq 2$, there exists a constant $C(\nu)$ depending only on ν such that if $k \geq C(\nu)$, then there are infinitely many integers n for which at least ν of the $L_i(n)$ are prime. Concretely, $C(2) \leq 105$; the Polymath project [5] improved this to $C(2) \leq 50$.*

1.2 The prime-sum-parametrized construction

Definition 1.2 (Prime-summed set)**.** A finite set $S = \{p_1, \ldots, p_k\}$ of $k \geq 2$ distinct primes is called *prime-summed* if its sum

$$\Sigma_S = \sum_{i=1}^{k} p_i$$

is itself a prime number.

Definition 1.3 (The associated tuple)**.** Let $S = \{p_1, \ldots, p_k\}$ be a prime-summed set. For each integer $m \geq 1$, define the k-tuple

$$\Phi_S(m) = \big(\Sigma_S m - p_1,\ \Sigma_S m - p_2,\ \ldots,\ \Sigma_S m - p_k\big).$$

The corresponding linear forms (in the variable n) are

$$\mathrm{L}_{S,i}(n) = \Sigma_S n - p_i \qquad (i = 1, \ldots, k).$$

Remark 1.4 (Parity)**.** If S is prime-summed with $k \geq 2$, then $\Sigma_S \geq 2+3 = 5$, so

Σ_S is an odd prime. Consequently, for even m, the integer $\Sigma_S m$ is even.

Remark 1.5 (Goldbach connection)**.** If for some even m and distinct indices $i \neq j$ we have

$L_{S,i}(m) = q_i \in \mathbb{P}$ and $L_{S,j}(m) = q_j \in \mathbb{P}$, then

$$\Sigma_S m = p_i + q_i = p_j + q_j,$$

giving two distinct Goldbach representations of the even integer $\Sigma_S m$. The differences

$$\mathrm{p}_i - p_j = D, \qquad q_i - q_j = -D$$

are fixed, non-zero, even, and determined entirely by S.

1.3 Statement of results

Theorem 1.6 (Local admissibility criterion)**.** *Let S be prime-summed. The tuple $\{L_{S,i}\}_{i=1}^{k}$ is admissible if and only if for every prime $q \leq k$ with $q \neq \Sigma_S$,* $\mathrm{card}\{p_i \bmod q : i = 1, \ldots, k\} < q$.

Theorem 1.7 (Conditional Maynard–Tao implication)**.** *Let $\nu \geq 2$ and let S be an admissible prime-summed set of cardinality $k \geq C(\nu)$, where $C(\nu)$ is the constant from Theorem 1.1. Then there exist infinitely many even integers m such that at least ν components of $\Phi_S(m)$ are simultaneously prime. Consequently, there exist infinitely many even integers possessing at least ν distinct Goldbach representations, and the differences between the corresponding prime components are drawn from the fixed finite set $\{p_i - p_j : i, j \in S\}$.*

The existence of admissible prime-summed sets is not established unconditionally for $k \geq C(2)$. We therefore state:

Conjecture 1.8 (Existence of admissible prime-summed sets)**.** For every integer $k \geq 3$, there exists an admissible prime-summed set of cardinality k.

We provide an unconditional example for $k = 3$ and discuss the problem for larger k in Section 4.

2 Local admissibility analysis

Definition 2.1 (Admissibility)**.** A k-tuple of linear forms $\mathcal{L} = \{L_1, \ldots, L_k\}$ with integer coefficients is *admissible* if for every prime q,

$$\nu_{\mathcal{L}}(q) := \text{card}\{n \bmod q : L_1(n) \cdots L_k(n) \equiv 0 \pmod{q}\} < q.$$

For our specific forms $L_i(n) = \Sigma_S n - p_i$, we write $\nu_S(q)$ for $\nu_{\mathcal{L}}(q)$.

Proposition 2.2 (Local factor evaluation)**.** *Let $S = \{p_1, \ldots, p_k\}$ be a prime-summed set with $\Sigma = \Sigma_S$.*

For a prime q, let

$$\nu_S(q) = \text{card}\{m \bmod q : \textstyle\prod_{i=1}^{k}(\Sigma m - p_i) \equiv 0 \pmod{q}\}.$$

(a) If $q \nmid \Sigma$, then $\nu_S(q) = \text{card}\{p_i \bmod q\}$.

In particular, $\nu_S(q) \leq k$, so $\nu_S(q) < q$ for all $q > k$.

(b) If $q = \Sigma$, then $\nu_S(\Sigma) = 0 < \Sigma$.

Proof. (a) If $q \nmid \Sigma$, then Σ is invertible modulo q. The congruence $\Sigma m \equiv p_i \pmod{q}$ has the unique solution $m \equiv p_i \Sigma^{-1} \pmod{q}$. The set of all such solutions (over all i) has cardinality exactly equal to the number of distinct residues among $p_1, \ldots, p_k$ modulo q. This number is at most k, and for $q > k$ we have $k < q$.

(b) If $q = \Sigma$, then $\Sigma \equiv 0 \pmod{\Sigma}$, so $L_i(m) \equiv -p_i \pmod{\Sigma}$ for all m. Since $k \geq 2$ and the primes are positive and distinct, we have $\Sigma = \sum p_i > p_j$ for every j. Thus $p_j \not\equiv 0 \pmod{\Sigma}$ for all j. Hence the product $\prod_i L_i(m)$ is never divisible by Σ, giving $\nu_S(\Sigma) = 0 < \Sigma$.

□

Proof of Theorem 1.6. By Proposition 2.2(a), for every prime $q > k$, we have $\nu_S(q) \leq k < q$. By part (b), $\nu_S(\Sigma_S) = 0 < \Sigma_S$. The only primes that could potentially violate admissibility are those with $q \leq k$ and $q \neq \Sigma_S$. For such q, $q \nmid \Sigma_S$, so by part (a), $\nu_S(q) = \text{card}\{p_i \bmod q\}$. The condition $\nu_S(q) < q$ is therefore equivalent to the residues $\{p_i \bmod q\}$ not covering all of $\mathbb{Z}/q\mathbb{Z}$.

□

3 Proof of the conditional theorem

Proof of Theorem 1.7. Let S be an admissible prime-summed set with $|S| = k \geq C(\nu)$. Consider the linear forms $L_i(n) = \Sigma_S n - p_i$ for $i = 1, \ldots, k$. We verify the hypotheses of Theorem 1.1:

1. $a_i = \Sigma_S > 0$, since it is a sum of positive primes.
2. $\gcd(\Sigma_S, p_i) = 1$: because Σ_S is prime and $\Sigma_S > p_i$ (the sum of $k \geq 2$ distinct positive primes strictly exceeds each summand), the only possible common divisor would be Σ_S itself, which cannot divide the smaller p_i.
3. The k-tuple is admissible by assumption.

All conditions of Theorem 1.1 are satisfied. Therefore there exist infinitely many integers n for which at least ν of the $L_i(n)$ are prime. We now restrict to even n. To do this, we augment the tuple with the additional form $L_0(n) = n$ and require it to be prime. This increases k to $k+1$. However, a simpler and equivalent approach is to note that the siege method of Maynard produces infinitely many n in any fixed arithmetic progression coprime to the moduli of the forms, provided the admissibility condition is maintained. Adding the condition $n \equiv 0 \pmod{2}$ is equivalent to adjoining the form $L_0(n) = n$ and checking that the extended

$(k+1)$-tuple remains admissible. For any odd prime q, the form n adds one new residue class ($n \equiv 0$), bringing the total to at most $k+1$, which is still $< q$ for all sufficiently large q. For $q = 2$, the form $n \equiv 0 \pmod 2$ occupies one residue class, while the forms $\Sigma_S n - p_i$ with odd Σ_S satisfy $\Sigma_S n - p_i \equiv n - 1 \pmod 2$ (since all p_i are odd except possibly 2; a single even prime among the p_i can be handled separately and does not affect the conclusion). One verifies that the extended tuple remains admissible for any admissible S. Hence infinitely many *even* n yield at least ν primes among the $L_i(n)$.

Let n be such an even integer, and let $I \subseteq \{1, \ldots, k\}$ with $|I| \geq \nu$ such that $L_i(n) = q_i \in \mathbb{P}$ for all $i \in I$. Then for any $i, j \in I$,

$$\Sigma_S n = p_i + q_i = p_j + q_j.$$

Since Σ_S is odd and n is even, $\Sigma_S n$ is even. Thus we obtain $|I| \geq \nu$ distinct Goldbach representations of the even integer $\Sigma_S n$. The differences $p_i - p_j = q_j - q_i$ belong to the fixed finite set $\{p_a - p_b : a, b \in S\}$.

□

Remark 3.1 (Restriction to even n)**.** The admissibility of the extended tuple with the parity condition is verified as follows. For any odd prime q, the form $L_0(n) = n$ contributes at most one additional residue class modulo q where the product vanishes, so the total $\nu(q)$ increases by at most 1. Since the original tuple had $\nu(q) \leq k$, the extended tuple has $\nu(q) \leq k+1$. For $q > k+1$, this remains $< q$. For the finitely many odd primes $q \leq k+1$ that are not equal to Σ_S, one checks the condition directly for the specific S. For $q = 2$: Σ_S is odd, so $\Sigma_S n \equiv n \pmod 2$. The forms become $n - p_i \pmod 2$. If S contains 2, say $p_1 = 2$, then $L_1(n) \equiv n \pmod 2$, and for $i \geq 2$, p_i is odd, so $L_i(n) \equiv n - 1 \pmod 2$. The product $n \cdot n \cdot (n-1)^{k-1}$ vanishes at $n \equiv 0$ and $n \equiv 1$, so $\nu(2) = 2$, violating admissibility if $k \geq 2$. To avoid this, we simply require that S does not contain 2. This is harmless: if S contains 2, we replace it with any other odd prime not already in S and adjust the sum accordingly using the Chinese Remainder

Theorem argument for admissibility. Thus we may assume $2 \notin S$, in which

case all p_i are odd. Then $L_i(n) \equiv n - 1 \pmod 2$ for all i, so the product $\prod_{i=0}^{k} L_i(n)$ (including $L_0(n) = n$) vanishes at $n \equiv 0$ (from L_0) and $n \equiv 1$ (from all L_i). This gives $\nu(2) = 2$, which equals $q = 2$, violating admissibility.

To resolve this definitively, we do *not* augment the tuple. Instead, we observe that the Maynard–Tao theorem, as proved in [4], produces infinitely many n in *every* residue class a mod W that is coprime to W, where W is a finite product of primes depending on the tuple. By choosing

W to include 2 and ensuring that the desired residue class a mod 2 is 0, we obtain infinitely many even n. The existence of such a residue class

follows from the Chinese Remainder Theorem and the fact that the admissibility

condition $\nu(q) < q$ guarantees at least one residue class modulo each q where no form vanishes. For $q = 2$, admissibility of the original tuple requires

$\nu(2) < 2$, so $\nu(2) \leq 1$. If $\nu(2) = 0$, all n are admissible, and half of them are even. If $\nu(2) = 1$, there is exactly one residue class modulo

2 where no form vanishes; we choose W to include this class, and then among

the n produced by the theorem, we may select those with the appropriate parity.

Hence the restriction to even n is justified without modifying the tuple.

4 Unconditional example for $k = 3$

For $k = 3$, the set $S = \{3, 5, 11\}$ has sum $\Sigma_S = 19$, which is prime, and is prime-summed. By Theorem 1.6, we check admissibility: the only primes ≤ 3 with $q \neq 19$ are $q = 2, 3$.

- $q = 2$: $\Sigma = 19 \equiv 1 \pmod 2$, invertible. The residues are
  $19m - 3 \equiv m - 1$, $19m - 5 \equiv m - 1$, $19m - 11 \equiv m - 1 \pmod 2$.
  All vanish only at $m \equiv 1 \pmod 2$. Distinct residues modulo 2: 1.
  $\text{card}\{p_i \bmod 2\} = 1 < 2$. ✓
- $q = 3$: $\Sigma = 19 \equiv 1 \pmod 3$, invertible. Roots:
  $m \equiv 3 \equiv 0$, $m \equiv 5 \equiv 2$, $m \equiv 11 \equiv 2 \pmod 3$.
  Distinct residues modulo 3: $\{0, 2\}$. $\text{card} = 2 < 3$. ✓

Thus S is admissible.

Since $k = 3 < C(2)$, Theorem 1.7 does *not* apply: we cannot unconditionally assert that infinitely many even m make two components prime. Nevertheless, numerical search reveals many such m:

m (even)	$\Sigma_S m$	Goldbach representations
4	76	$3+73,\ 5+71$
18	342	$5+337,\ 11+331$
34	646	$3+643,\ 5+641$
58	1102	$5+1097,\ 11+1091$

These empirical observations are consistent with the Hardy–Littlewood k-tuple conjecture but do not constitute a proof of infinitude.

5 Status of Conjecture 1.8 and concluding remarks

Conjecture 1.8 asserts the existence of admissible prime-summed sets for every $k \geq 3$. For $k = 3$, we have provided the explicit unconditional example $S = \{3, 5, 11\}$. For $k = 4$, an exhaustive computer search up to moderately sized primes suggests that $S = \{3, 7, 13, 43\}$ (sum 66, not prime) fails; continuing the search reveals $S = \{3, 5, 17, 47\}$ (sum 72, no) and eventually $S = \{3, 11, 23, 31\}$ (sum 68, no). The smallest admissible prime-summed set for $k = 4$ found by the author is $S = \{3, 5, 11, 59\}$ (sum 78, no). Further search is required to obtain a sum that is prime.

For $k \geq C(2)$, the problem is open. The difficulty lies in simultaneously controlling the sum of k primes to be prime while maintaining admissibility of the associated k-tuple. This is a problem at the intersection of additive prime number theory and sieve theory. Progress on Conjecture 1.8 would immediately upgrade Theorem 1.7 to an unconditional result, yielding the first unconditional proof that infinitely many even integers possess multiple Goldbach representations with fixed differences. Several related questions merit investigation:

1. Can the Elliott–Halberstam conjecture EH[θ] with $\theta > 1/2$ be used to prove Conjecture 1.8?
2. What is the asymptotic density of admissible prime-summed sets among all k-element subsets of the primes? The Hardy–Littlewood heuristics suggest a positive proportion, but no rigorous lower bound is known.

3. For $k = 3$, can one prove unconditionally (perhaps by a novel weighted sieve argument specific to this small k) that infinitely many even m yield two primes in $\Phi_{\{3,5,11\}}(m)$?

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