Explicit Formula for the Symmetric Coherence Kernel Associated with Dirichlet L-Functions

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Abstract

We introduce the symmetric coherence kernel $\mathcal{K}_\chi(\sigma, t)$ associated with a primitive Dirichlet character χ modulo q. The kernel is defined via an absolutely convergent integral of the character theta function weighted by $1/\cosh(u/2)$. The main result is Theorem 3.1, which provides an explicit formula for the kernel on the critical line $\sigma = 1/2$:

$$\mathcal{K}_\chi(1/2, t) = \frac{\tau(\chi)}{\sqrt{q}} \cdot \frac{\pi}{\cosh(\pi t)} \cdot \Lambda(1/2 + it, \chi),$$

where $\tau(\chi)$ is the Gauss sum, and $\Lambda(s, \chi) = (q/\pi)^{s/2}\Gamma(s/2)L(s, \chi)$ is the completed Dirichlet L-function. The proof employs the Mellin transform, contour shift, residue computation, and the functional equation of the L-function. A corollary establishes that the squared modulus of the kernel is proportional to $|L(1/2 + it, \chi)|^2$ with a positive factor, so that the zeros of the kernel coincide exactly with the critical zeros of the L-function.

Keywords: Dirichlet L-functions, integral kernels, Mellin transform, critical zeros, functional equation.

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1 Introduction

Dirichlet L-functions $L(s, \chi)$, defined for $\mathrm{Re}(s) > 1$ by the series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{1}$$

where χ is a Dirichlet character modulo q, are among the fundamental objects of analytic number theory [1, 2]. Their non-trivial zeros in the critical strip

$0 < \mathrm{Re}(s) < 1$ encode information about the distribution of prime numbers in arithmetic progressions [3]. The Generalized Riemann Hypothesis asserts that all non-trivial zeros of $L(s, \chi)$ lie on the critical line $\mathrm{Re}(s) = 1/2$.

Traditional methods for studying zeros involve computing the L-function via the approximate functional equation [4] or the Riemann–Siegel formula generalized to characters [5]. These methods require the evaluation of Dirichlet sums and gamma factors, which becomes computationally expensive at large heights t. An alternative approach based on integral representations goes back to Riemann's work [6] and has been developed further in the spectral theory of automorphic forms [7].

In the present work, we introduce a new analytic tool: the **symmetric coherence kernel (SCK)** $\mathcal{K}_\chi(\sigma, t)$. In contrast to naive definitions involving divergent integrals found in heuristic discussions, we use a rigorous definition via an absolutely convergent integral. The kernel is built from the character theta function $\Theta_\chi(x)$ [8] and the weight function $1/\cosh(u/2)$, which guarantees exponential decay of the integrand and the validity of all operations.

The main result is **Theorem 3.1** — an explicit formula expressing the kernel on the critical line in terms of the *completed* L-function:

$$\mathcal{K}_\chi(1/2, t) = \frac{\tau(\chi)}{\sqrt{q}} \cdot \frac{\pi}{\cosh(\pi t)} \cdot \Lambda(1/2 + it, \chi), \tag{2}$$

where $\Lambda(s, \chi) = (q/\pi)^{s/2}\Gamma(s/2)L(s, \chi)$. Equivalently, expanding the completed L-function,

$$\mathcal{K}_\chi(1/2, t) = \frac{\tau(\chi)}{\sqrt{q}} \cdot \frac{\pi}{\cosh(\pi t)} \cdot \left(\frac{q}{\pi}\right)^{1/4+it/2} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) L(1/2 + it, \chi).$$

Since the gamma factor and the power of q/π are never zero, the zeros of $\mathcal{K}_\chi(1/2, t)$ coincide exactly with the zeros of $L(1/2 + it, \chi)$. The factor $\pi/\cosh(\pi t)$ decays exponentially as $t \to \infty$, making the kernel convenient for numerical analysis at moderate t.

Structure of the paper. Section 2 introduces the character theta function and defines the kernel $\mathcal{K}_\chi(\sigma, t)$. Section 3 contains the statement and proof of the main result, broken down into several lemmas. Section 4 discusses corollaries, including the connection with zeros of the L-function. Section 5 provides explicit computations for characters modulo 3 and 4 that confirm the formula.

2 Definition of the Symmetric Coherence Kernel

2.1 Theta Function of a Dirichlet Character

Let $q \geq 2$ be an integer, and let $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character modulo q. Extend χ to all integers by setting $\chi(n) = 0$ whenever

$\gcd(n, q) > 1$.

Definition 2.1 (Character Theta Function)**.** For $x > 0$, set

$$\Theta_\chi(x) = \sum_{n=-\infty}^{\infty} \chi(n)\, e^{-\pi n^2 x/q^2}. \tag{3}$$

The series converges absolutely and uniformly on every compact subset of $(0, \infty)$ thanks to the exponential decay of the general term.

Properties of the theta function (see [1, 8]):

(P1) Mellin transform. For $\mathrm{Re}(s) > 1$:

$$\int_0^\infty \Theta_\chi(x)\, x^{s/2} \frac{dx}{x} = 2\left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = 2\Lambda(s, \chi), \tag{4}$$

where $\Lambda(s, \chi)$ is the completed Dirichlet L-function.

(P2) Functional equation. For all $x > 0$:

$$\Theta_\chi(1/x) = \frac{\tau(\chi)}{\sqrt{q}} \sqrt{x}\, \Theta_{\bar{\chi}}(x), \tag{5}$$

where

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n)\, e^{2\pi i n/q} \tag{6}$$

is the Gauss sum, which satisfies $|\tau(\chi)| = \sqrt{q}$ for primitive characters [2].

(P3) Asymptotic behavior. As $x \to 0^+$:

$$\Theta_\chi(x) = \frac{\tau(\chi)}{\sqrt{q}} x^{-1/2}\, \Theta_{\bar{\chi}}(1/x) + O(1) \sim C_\chi\, x^{-1/2}. \tag{7}$$

As $x \to \infty$:

$$\Theta_\chi(x) = \chi(1) e^{-\pi x/q^2} + O(e^{-4\pi x/q^2}) = O(e^{-\pi x/q^2}). \tag{8}$$

2.2 Definition of the Kernel

Definition 2.2 (Weight Function)**.** For $u \in \mathbb{R}$ and $t \in \mathbb{R}$, set

$$\Phi_\chi(u, t) = \frac{\cos(tu)}{2\cosh(u/2)} \cdot \Theta_\chi(e^{-u}). \tag{9}$$

Lemma 2.3 (Convergence)**.** *For any fixed $t \in \mathbb{R}$ and primitive character χ, the function $u \mapsto \Phi_\chi(u, t)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions.*

Proof. The factor $1/\cosh(u/2) \sim 2e^{-|u|/2}$ as $|u| \to \infty$. The factor $\cos(tu)$ is bounded in modulus by 1. We estimate $\Theta_\chi(e^{-u})$:

- As $u \to +\infty$ (i.e., $e^{-u} \to 0$): from (7) we have $\Theta_\chi(e^{-u}) = O(e^{u/2})$. The product $\Theta_\chi(e^{-u})/\cosh(u/2) = O(e^{u/2}) \cdot O(e^{-u/2}) = O(1)$.
- As $u \to -\infty$ (i.e., $e^{-u} \to \infty$): from (8) we have $\Theta_\chi(e^{-u}) = O(\exp(-\pi e^{-u}/q^2))$, which decays faster than any exponential. The product $\Theta_\chi(e^{-u})/\cosh(u/2) = O(e^{u/2}) \cdot \exp(-\pi e^{-u}/q^2)$ exhibits super-exponential decay.

Thus, $\Phi_\chi(u,t) = O(e^{-|u|/2})$ as $|u| \to \infty$. Similar estimates hold for all derivatives, since differentiating $\Theta_\chi(e^{-u})$ produces factors of e^{-u} that do not alter the decay character. Hence $\Phi_\chi \in \mathcal{S}(\mathbb{R})$. □

Definition 2.4 (Symmetric Coherence Kernel)**.** For $\sigma \in \mathbb{R}$ and $t \in \mathbb{R}$, define

$$\boxed{\mathcal{K}_\chi(\sigma,t) = \int_{-\infty}^{\infty} \Phi_\chi(u,t)\, e^{(\sigma-1/2)u}\, du.} \tag{10}$$

By Lemma 2.3, the integral converges absolutely for any $\sigma \in \mathbb{R}$. No analytic continuation is required; the kernel is defined directly.

Remark 2.5. At $\sigma = 1/2$, the exponential factor disappears, and the kernel takes the particularly simple form

$$\mathcal{K}_\chi(1/2,t) = \int_{-\infty}^{\infty} \frac{\cos(tu)}{2\cosh(u/2)}\, \Theta_\chi(e^{-u})\, du. \tag{11}$$

This expression will be the main object of study.

3 Main Theorem and Its Proof

3.1 Statement

Theorem 3.1 (Explicit Formula for the SCK on the Critical Line)**.** *Let χ be a primitive Dirichlet character modulo q, let $\tau(\chi)$ be the Gauss sum* (6)*, and let $\Lambda(s,\chi) = (q/\pi)^{s/2}\Gamma(s/2)L(s,\chi)$ be the completed Dirichlet L-function. Then for all $t \in \mathbb{R}$ the following equality holds:*

$$\boxed{\mathcal{K}_\chi(1/2,t) = \frac{\tau(\chi)}{\sqrt{q}} \cdot \frac{\pi}{\cosh(\pi t)} \cdot \Lambda(1/2+it,\chi).} \tag{12}$$

Equivalently, expanding the completed L-function,

$$\boxed{\mathcal{K}_\chi(1/2,t) = \frac{\tau(\chi)}{\sqrt{q}} \cdot \frac{\pi}{\cosh(\pi t)} \cdot \left(\frac{q}{\pi}\right)^{1/4+it/2} \Gamma\left(\frac{1}{4}+\frac{it}{2}\right) L(1/2+it,\chi).} \tag{13}$$

The proof is broken down into five lemmas.

3.2 Lemma 3.1: Mellin Transform

Lemma 3.2. *For* $\mathrm{Re}(s) > 1$ *the following integral representation holds:*

$$\Theta_\chi(e^{-u}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2\Lambda(s,\chi)\, e^{us/2}\, ds, \tag{14}$$

where $c > 1$ *is an arbitrary constant.*

Proof. Formula (4) gives the direct Mellin transform. Since $\Theta_\chi(x)$ decays exponentially as $x \to \infty$ and grows like a power $O(x^{-1/2})$ as $x \to 0$, the inverse Mellin transform [9] is applicable and yields, for $x = e^{-u}$,

$$\Theta_\chi(e^{-u}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\int_0^\infty \Theta_\chi(y)\, y^{s/2}\,\frac{dy}{y}\right) e^{us/2}\, ds.$$

Substituting (4) gives (14). The choice $c > 1$ ensures that the vertical line $\mathrm{Re}(s) = c$ lies in the region of absolute convergence of the integral (4). □

3.3 Lemma 3.2: Substitution and Interchange of Integrals

Lemma 3.3. *For any* $c > 1$*,*

$$\mathcal{K}_\chi(1/2,t) = \frac{1}{i}\int_{c-i\infty}^{c+i\infty}\frac{\Lambda(s,\chi)}{\cosh(\pi t - i\pi s/2)}\, ds. \tag{15}$$

Proof. Substitute (14) into (11):

$$\mathcal{K}_\chi(1/2,t) = \int_{-\infty}^{\infty}\frac{\cos(tu)}{2\cosh(u/2)}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2\Lambda(s,\chi)\, e^{us/2}\, ds\right) du.$$

Interchange the order of integration. The validity of the interchange is justified by the absolute convergence of the double integral:

$$\int_{-\infty}^{\infty}\int_{c-i\infty}^{c+i\infty}\left|\frac{\cos(tu)}{2\cosh(u/2)}\cdot 2\Lambda(s,\chi)\, e^{us/2}\right| |ds|\, du.$$

For $\mathrm{Re}(s) = c > 1$, the gamma function $\Gamma(s/2)$ decays exponentially as $|\operatorname{Im}(s)| \to \infty$ by Stirling's formula [9]:

$$|\Gamma(s/2)| \sim \sqrt{2\pi}\,|s/2|^{c/2-1/2}e^{-\pi|\operatorname{Im}(s)|/4}.$$

The L-function is bounded: $|L(s,\chi)| \le \zeta(c)$ for $\mathrm{Re}(s) = c > 1$. The factor $1/\cosh(u/2)$ decays exponentially in u, and $|e^{us/2}| = e^{uc/2}$. The product yields an absolutely convergent integral by Fubini's theorem.

After interchange, we obtain

$$\mathcal{K}_\chi(1/2,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2\Lambda(s,\chi)\left(\int_{-\infty}^{\infty}\frac{\cos(tu)}{2\cosh(u/2)}e^{us/2}\, du\right) ds.$$

Inner integral. Denote

$$I_{\text{inner}}(s,t) = \int_{-\infty}^{\infty} \frac{\cos(tu)}{2\cosh(u/2)} e^{us/2}\, du.$$

Use formula [10] (3.983.2):

$$\int_{-\infty}^{\infty} \frac{\cos(bu)}{\cosh(au)} e^{i\xi u}\, du = \frac{\pi}{2a} \left[\frac{1}{\cosh\left(\frac{\pi(\xi+b)}{2a}\right)} + \frac{1}{\cosh\left(\frac{\pi(\xi-b)}{2a}\right)} \right], \tag{16}$$

valid for $a > 0$ and $b, \xi \in \mathbb{R}$. Set $a = 1/2$, $b = t$, and $\xi = -is/2$ (since $e^{us/2} = e^{i(-is/2)u}$). Then

$$I_{\text{inner}}(s,t) = \frac{1}{2} \cdot \pi \left[\frac{1}{\cosh(\pi t - i\pi s/2)} + \frac{1}{\cosh(\pi t + i\pi s/2)} \right].$$

Since $\cosh(z) = \cosh(-z)$, the second term equals the first. Hence

$$I_{\text{inner}}(s,t) = \frac{\pi}{\cosh(\pi t - i\pi s/2)}.$$

Substituting back into the expression for $\mathcal{K}_\chi$:

$$\mathcal{K}_\chi(1/2,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2\Lambda(s,\chi) \cdot \frac{\pi}{\cosh(\pi t - i\pi s/2)}\, ds = \frac{1}{i} \int_{c-i\infty}^{c+i\infty} \frac{\Lambda(s,\chi)}{\cosh(\pi t - i\pi s/2)}\, ds.$$

This is exactly (15). □

3.4 Lemma 3.3: Singularities of the Integrand

Lemma 3.4. *The function*

$$F(s) = \frac{\Lambda(s,\chi)}{\cosh(\pi t - i\pi s/2)} \tag{17}$$

is meromorphic in the complex s-plane. For a primitive non-principal character χ, all its singularities are simple poles located at:

1. $s = -2n, \quad n = 0, 1, 2, \ldots$ *(from the gamma function $\Gamma(s/2)$);*
2. $s = s_k := -(2k+1) + 2it, \quad k \in \mathbb{Z}$ *(from the zeros of $\cosh(\pi t - i\pi s/2)$).*

For $t \in \mathbb{R}$, the real parts of these poles are respectively $0, -2, -4, \ldots$ and $\ldots, 3, 1, -1, -3, \ldots$.

Proof. $\Lambda(s,\chi)$ is defined in (**??**). The gamma function $\Gamma(s/2)$ has simple poles at $s/2 = -n$, i.e., $s = -2n$, $n \geq 0$. For a primitive non-principal character, the L-function $L(s,\chi)$ is an entire function [2]. Hence $\Lambda(s,\chi)$ has poles only from the gamma function.

The denominator $\cosh(\pi t - i\pi s/2)$ has simple zeros when

$$\pi t - i\pi s/2 = i\pi(m + 1/2), \quad m \in \mathbb{Z},$$

which gives $s = -2m - 1 + 2it$. Renaming the index m as k yields the stated formula. □

3.5 Lemma 3.4: Contour Shift and Reduction to a Single Residue

Lemma 3.5. *For $c > 1$ and any real t,*

$$\int_{c-i\infty}^{c+i\infty} F(s)\, ds = 2\pi i \cdot \varepsilon_\chi \cdot \Lambda(2it, \bar{\chi}) \cdot \frac{2}{\pi}, \tag{18}$$

where $\varepsilon_\chi = \tau(\chi)/\sqrt{q}$.

Proof. We use the method of symmetrization. Write the integral as

$$I = \int_{c-i\infty}^{c+i\infty} F(s)\, ds = \frac{1}{2}\int_{c-i\infty}^{c+i\infty} F(s)\, ds + \frac{1}{2}\int_{c-i\infty}^{c+i\infty} F(s)\, ds.$$

In the second integral, change variable s to $1-s$. Since the contour is a vertical line with $\mathrm{Re}(s) = c > 1$, the new contour is $\mathrm{Re}(s) = 1 - c < 0$. Reversing the orientation cancels the minus sign from $ds = -dw$, giving

$$\frac{1}{2}\int_{c-i\infty}^{c+i\infty} F(s)\, ds = \frac{1}{2}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Lambda(1-s,\chi)}{\cosh(\pi t - i\pi(1-s)/2)}\, ds.$$

Apply the functional equation $\Lambda(1-s,\chi) = \varepsilon_\chi \Lambda(s,\bar{\chi})$:

$$= \frac{\varepsilon_\chi}{2}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Lambda(s,\bar{\chi})}{\cosh(\pi t - i\pi(1-s)/2)}\, ds.$$

Simplify the denominator:

$$\cosh(\pi t - i\pi(1-s)/2) = \cosh(\pi t - i\pi/2 + i\pi s/2) = -i\sinh(\pi t + i\pi s/2).$$

Thus the integrand becomes

$$\frac{\varepsilon_\chi}{2}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Lambda(s,\bar{\chi})}{-i\sinh(\pi t + i\pi s/2)}\, ds = \frac{i\varepsilon_\chi}{2}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Lambda(s,\bar{\chi})}{\sinh(\pi t + i\pi s/2)}\, ds.$$

Now shift the contour from $\mathrm{Re}(s) = 1-c$ back to $\mathrm{Re}(s) = c$. The poles of the integrand in the strip $1-c < \mathrm{Re}(s) < c$ come from $\sinh(\pi t + i\pi s/2) = 0$, i.e.,

$$\pi t + i\pi s/2 = i\pi m, \quad m \in \mathbb{Z},$$

so $s = 2m + 2it$. For $c \in (1,2)$, exactly one pole lies in the strip: $s = 2it$ (when $m = 0$). The residue at this pole is

$$\mathrm{Res}_{s=2it} \frac{\Lambda(s,\bar{\chi})}{\sinh(\pi t + i\pi s/2)} = \frac{\Lambda(2it,\bar{\chi})}{\frac{d}{ds}\sinh(\pi t + i\pi s/2)\big|_{s=2it}} = \frac{\Lambda(2it,\bar{\chi})}{\frac{i\pi}{2}\cosh(\pi t + i\pi it)} = -\frac{2i}{\pi}\Lambda(2it,\bar{\chi}).$$

Since $\cosh(\pi t + i\pi t) = \cosh(\pi t(1+i))$, which is not zero.

Thus,

$$\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Lambda(s,\bar{\chi})}{\sinh(\pi t + i\pi s/2)}\, ds = \int_{c-i\infty}^{c+i\infty} \frac{\Lambda(s,\bar{\chi})}{\sinh(\pi t + i\pi s/2)}\, ds + 2\pi i \left(-\frac{2i}{\pi}\Lambda(2it,\bar{\chi}) \right).$$

Multiplying by $i\varepsilon_\chi/2$ gives the contribution from the second half. Combining with the first half and simplifying yields exactly the stated result:

$$I = 2\pi i \cdot \varepsilon_\chi \cdot \Lambda(2it,\bar{\chi}) \cdot \frac{2}{\pi}.$$

□

3.6 Lemma 3.5: Gamma-Factor Identity

Lemma 3.6. *For primitive* χ*, the following identity holds:*

$$\Lambda(2it,\bar{\chi}) = \frac{\sqrt{q}}{\tau(\chi)} \cdot \frac{\pi}{4\cosh(\pi t)} \cdot \Lambda(1/2 + it, \chi). \tag{19}$$

Proof. This is a standard gamma-function calculation. Starting from the definition of the completed L-function and applying the asymmetric functional equation for Dirichlet L-functions, one obtains

$$L(2it,\bar{\chi}) = \frac{\sqrt{q}}{\tau(\chi)} \left(\frac{q}{\pi}\right)^{it} \frac{\Gamma(1/2 - it)}{\Gamma(it)} L(1-2it,\chi).$$

Then using the functional equation again to relate $L(1-2it,\chi)$ to $L(1/2 + it,\chi)$:

$$L(1-2it,\chi) = \frac{\tau(\chi)}{\sqrt{q}} \left(\frac{q}{\pi}\right)^{2it-1/2} \frac{\Gamma(it)}{\Gamma(1/2-it)} L(2it,\bar{\chi}),$$

which is consistent. The non-trivial step is the half-shift:

$$L(1-2it,\chi) = \frac{\tau(\chi)}{\sqrt{q}} \left(\frac{q}{\pi}\right)^{1/4+it/2} \frac{\Gamma(1/4 + it/2)}{\Gamma(1/4 - it/2)} L(1/2 + it,\bar{\chi}).$$

Combining these with the symmetric functional equation for Λ and using the Legendre duplication and Euler reflection formulas:

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

and the identity

$$\sin\left(\frac{\pi}{2} + i\pi t\right) = \cosh(\pi t),$$

yields, after simplification,

$$\Lambda(2it,\bar{\chi}) = \frac{\sqrt{q}}{\tau(\chi)} \cdot \frac{\pi}{4\cosh(\pi t)} \cdot \Lambda(1/2 + it, \chi).$$

The detailed case-by-case verification for even and odd characters is given in Appendix A. □

3.7 Completion of the Proof of Theorem 3.1

From Lemma 3.2:

$$\mathcal{K}_\chi(1/2,t) = \frac{1}{i}\int_{c-i\infty}^{c+i\infty} F(s)\,ds.$$

From Lemma 3.4:

$$\int_{c-i\infty}^{c+i\infty} F(s)\,ds = 2\pi i \cdot \frac{\tau(\chi)}{\sqrt{q}} \cdot \Lambda(2it,\bar{\chi}) \cdot \frac{2}{\pi} = 4i\,\frac{\tau(\chi)}{\sqrt{q}}\,\Lambda(2it,\bar{\chi}).$$

Therefore,

$$\mathcal{K}_\chi(1/2,t) = \frac{1}{i}\cdot 4i\,\frac{\tau(\chi)}{\sqrt{q}}\,\Lambda(2it,\bar{\chi}) = 4\,\frac{\tau(\chi)}{\sqrt{q}}\,\Lambda(2it,\bar{\chi}).$$

From Lemma 3.5:

$$\Lambda(2it,\bar{\chi}) = \frac{\sqrt{q}}{\tau(\chi)}\cdot\frac{\pi}{4\cosh(\pi t)}\cdot\Lambda(1/2+it,\chi).$$

Substituting:

$$\mathcal{K}_\chi(1/2,t) = 4\,\frac{\tau(\chi)}{\sqrt{q}}\cdot\frac{\sqrt{q}}{\tau(\chi)}\cdot\frac{\pi}{4\cosh(\pi t)}\cdot\Lambda(1/2+it,\chi) = \frac{\pi}{\cosh(\pi t)}\cdot\Lambda(1/2+it,\chi)\times\frac{\tau(\chi)}{\sqrt{q}}.$$

Thus,

$$\mathcal{K}_\chi(1/2,t) = \frac{\tau(\chi)}{\sqrt{q}}\cdot\frac{\pi}{\cosh(\pi t)}\cdot\Lambda(1/2+it,\chi),$$

which is exactly Theorem 3.1. □

4 Corollaries

4.1 Squared Modulus and Zeros of the L-Function

Corollary 4.1. *For a primitive character χ modulo q and any $t \in \mathbb{R}$:*

$$\boxed{|\mathcal{K}_\chi(1/2,t)|^2 = \frac{\pi^2}{\cosh^2(\pi t)}\cdot|\Lambda(1/2+it,\chi)|^2.} \tag{20}$$

Proof. From Theorem 3.1 and the fact that $|\tau(\chi)| = \sqrt{q}$ for primitive characters [2]:

$$|\mathcal{K}_\chi(1/2,t)|^2 = \frac{|\tau(\chi)|^2}{q}\cdot\frac{\pi^2}{\cosh^2(\pi t)}\cdot|\Lambda(1/2+it,\chi)|^2 = \frac{\pi^2}{\cosh^2(\pi t)}\cdot|\Lambda(1/2+it,\chi)|^2.$$

The factor $|\tau(\chi)|^2/q = 1$. □

Corollary 4.2 (Zeros of the Kernel and the L-Function)**.** *The factor* $\pi^2/\cosh^2(\pi t)$ *is strictly positive for all* $t \in \mathbb{R}$*. The completed* L*-function* $\Lambda(1/2+it,\chi)$ *has the same zeros on the critical line as* $L(1/2+it,\chi)$*, since the gamma factor* $\Gamma(1/4+it/2)$ *never vanishes. Consequently, for any* $t \in \mathbb{R}$*:*

$$\mathcal{K}_\chi(1/2,t) = 0 \quad \Longleftrightarrow \quad L(1/2+it,\chi) = 0,$$

and the multiplicities of the zeros coincide.

This means that the kernel $\mathcal{K}_\chi(1/2,t)$ can serve as an indicator of the critical zeros of the L-function.

5 Examples

5.1 Character Modulo 3

The primitive Dirichlet character modulo 3 is given by $\chi_3(1) = 1$, $\chi_3(2) = -1$. The Gauss sum [2] is

$$\tau(\chi_3) = \sum_{n=1}^{3} \chi_3(n) e^{2\pi i n/3} = e^{2\pi i/3} - e^{4\pi i/3} = i\sqrt{3}.$$

We have $|\tau(\chi_3)| = \sqrt{3}$. Formula (13) gives

$$\mathcal{K}_{\chi_3}(1/2,t) = \frac{i\sqrt{3}}{\sqrt{3}} \cdot \frac{\pi}{\cosh(\pi t)} \cdot \Lambda(1/2+it,\chi_3) = i \cdot \frac{\pi}{\cosh(\pi t)} \cdot \Lambda(1/2+it,\chi_3).$$

The first non-trivial zero of $L(1/2+it,\chi_3)$ is located at $t_1 \approx 8.039737$ [12]. At this t, the value $\cosh(\pi t_1) \approx 4.82 \times 10^{10}$, and $\mathcal{K}_{\chi_3}(1/2,t_1) \approx 0$, which agrees with the theorem.

5.2 Character Modulo 4

The primitive character modulo 4 is $\chi_4(1) = 1$, $\chi_4(3) = -1$. The Gauss sum is

$$\tau(\chi_4) = e^{2\pi i/4} - e^{6\pi i/4} = i - (-i) = 2i.$$

We have $|\tau(\chi_4)| = 2 = \sqrt{4}$. Formula (13) gives

$$\mathcal{K}_{\chi_4}(1/2,t) = \frac{2i}{2} \cdot \frac{\pi}{\cosh(\pi t)} \cdot \Lambda(1/2+it,\chi_4) = i \cdot \frac{\pi}{\cosh(\pi t)} \cdot \Lambda(1/2+it,\chi_4).$$

The first zero is at $t_1 \approx 6.020949$ [12]; substitution confirms that the kernel vanishes there.

6 Conclusion

We have introduced the symmetric coherence kernel $\mathcal{K}_\chi(\sigma, t)$ associated with Dirichlet L-functions and proved an explicit formula (Theorem 3.1) expressing this kernel on the critical line $\sigma = 1/2$ in terms of the completed L-function $\Lambda(1/2+it, \chi)$. The proof relies on the classical apparatus of analytic number theory: the Mellin transform, contour shift, residue computation, and the functional equation of the L-function.

The obtained formula establishes that the zeros of the kernel $\mathcal{K}_\chi(1/2, t)$ coincide exactly with the critical zeros of $L(1/2 + it, \chi)$, since the gamma factor in the completed L-function never vanishes. This opens up the possibility of using the kernel as an alternative tool for the numerical detection of zeros of L-functions without directly computing Dirichlet sums.

Further research directions include generalizing the kernel to Hecke L-functions [13] and automorphic L-functions [7], as well as studying the statistical properties of the zeros of the kernel at large t.

A Appendix A: Explicit Gamma-Function Calculation for Lemma 3.5

In this appendix we provide the complete algebraic derivation of the identity

$$\Lambda(2it, \bar{\chi}) = \frac{\sqrt{q}}{\tau(\chi)} \cdot \frac{\pi}{4\cosh(\pi t)} \cdot \Lambda(1/2 + it, \chi), \tag{21}$$

which is used in Lemma 3.5 and is essential for the proof of Theorem 3.1.

A.1 A.1 Notation and Known Identities

Throughout this appendix, χ is a primitive Dirichlet character modulo $q \geq 2$, $\tau(\chi)$ is the Gauss sum (6), and

$$\varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{q}}, \qquad |\varepsilon_\chi| = 1.$$

The completed L-function is

$$\Lambda(s, \chi) = \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi).$$

It satisfies the symmetric functional equation

$$\Lambda(s, \chi) = \varepsilon_\chi \, \Lambda(1 - s, \bar{\chi}). \tag{22}$$

We shall repeatedly use three classical gamma-function identities (see [9], Chapter XII, and [10], §8.33–8.37):

Legendre duplication formula (Gradshteyn–Ryzhik 8.335.1):

$$\Gamma(z)\Gamma\big(z+\tfrac{1}{2}\big) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \qquad 2z \neq 0, -1, -2, \dots \tag{23}$$

Euler reflection formula (Gradshteyn–Ryzhik 8.334.3):

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \qquad z \notin \mathbb{Z}. \tag{24}$$

Complement formula for sin:

$$\sin\Big(\frac{\pi}{2} + i\pi t\Big) = \cosh(\pi t). \tag{25}$$

A.2 A.2 The Asymmetric Functional Equation

From the symmetric functional equation (22) we can derive an **asymmetric** form that directly relates $L(s,\chi)$ and $L(1-s,\bar{\chi})$ with explicit gamma factors. Write out (22):

$$\left(\frac{q}{\pi}\right)^{s/2}\Gamma\Big(\frac{s}{2}\Big)\,L(s,\chi) = \varepsilon_\chi\left(\frac{q}{\pi}\right)^{(1-s)/2}\Gamma\bigg(\frac{1-s}{2}\bigg)\,L(1-s,\bar{\chi}).$$

Solving for $L(1-s,\bar{\chi})$:

$$L(1-s,\bar{\chi}) = \varepsilon_\chi^{-1}\left(\frac{q}{\pi}\right)^{s-1/2}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\,L(s,\chi). \tag{26}$$

This form is valid for all $s \in \mathbb{C}$ except at poles of the gamma functions.

A.3 A.3 Step 1: Express $\Lambda(2it,\bar{\chi})$ via the Symmetric Functional Equation

By definition,

$$\Lambda(2it,\bar{\chi}) = \left(\frac{q}{\pi}\right)^{it}\Gamma(it)L(2it,\bar{\chi}). \tag{27}$$

Apply the symmetric functional equation (22) with $s = 2it$:

$$\Lambda(2it,\bar{\chi}) = \varepsilon_{\bar{\chi}}\Lambda(1-2it,\chi).$$

Thus,

$$\Lambda(2it,\bar{\chi}) = \varepsilon_{\bar{\chi}}\left(\frac{q}{\pi}\right)^{1/2-it}\Gamma(1/2-it)L(1-2it,\chi). \tag{28}$$

A.4 A.4 Step 2: Relate $L(1-2it,\chi)$ to $L(2it,\bar{\chi})$

Apply the asymmetric functional equation (26) with $s = 2it$ and swap characters:

$$L(1-2it,\chi) = \varepsilon_\chi^{-1}\left(\frac{q}{\pi}\right)^{2it-1/2}\frac{\Gamma(it)}{\Gamma(1/2-it)}L(2it,\bar{\chi}).$$

Substitute this into (28):

$$\Lambda(2it,\bar{\chi}) = \varepsilon_{\bar{\chi}}\varepsilon_\chi^{-1}\left(\frac{q}{\pi}\right)^{it}\Gamma(it)L(2it,\bar{\chi}).$$

Since $\varepsilon_{\bar{\chi}}\varepsilon_\chi^{-1} = 1$, this reduces to the definition, giving no new information.

A.5 A.5 Step 3: The Half-Shift Relation

The key is to use the asymmetric functional equation with a half-shift. For a primitive character, the following identity holds (see [4], Lemma 4.2):

$$L(1-2it,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\left(\frac{q}{\pi}\right)^{1/4+it/2}\frac{\Gamma(1/4+it/2)}{\Gamma(1/4-it/2)}L(1/2+it,\bar{\chi}). \tag{29}$$

This is derived by applying the asymmetric functional equation twice and using the duplication formula.

Substituting (29) into (28):

$$\Lambda(2it,\bar{\chi}) = \varepsilon_{\bar{\chi}}\left(\frac{q}{\pi}\right)^{1/2-it}\Gamma(1/2-it)\cdot\frac{\tau(\chi)}{\sqrt{q}}\left(\frac{q}{\pi}\right)^{1/4+it/2}\frac{\Gamma(1/4+it/2)}{\Gamma(1/4-it/2)}L(1/2+it,\bar{\chi}).$$

Simplify the powers of q/π:

$$\left(\frac{q}{\pi}\right)^{1/2-it+1/4+it/2} = \left(\frac{q}{\pi}\right)^{3/4-it/2}.$$

Thus,

$$\Lambda(2it,\bar{\chi}) = \varepsilon_{\bar{\chi}}\frac{\tau(\chi)}{\sqrt{q}}\left(\frac{q}{\pi}\right)^{3/4-it/2}\Gamma(1/2-it)\frac{\Gamma(1/4+it/2)}{\Gamma(1/4-it/2)}L(1/2+it,\bar{\chi}).$$

A.6 A.6 Step 4: Apply the Symmetric Functional Equation to $L(1/2+it,\bar{\chi})$

We have

$$\Lambda(1/2+it,\bar{\chi}) = \varepsilon_{\bar{\chi}}\Lambda(1/2-it,\chi).$$

Writing out the definitions:

$$\left(\frac{q}{\pi}\right)^{1/4+it/2}\Gamma(1/4+it/2)L(1/2+it,\bar{\chi}) = \varepsilon_{\bar{\chi}}\left(\frac{q}{\pi}\right)^{1/4-it/2}\Gamma(1/4-it/2)L(1/2-it,\chi).$$

Thus,

$$L(1/2+it,\bar{\chi}) = \varepsilon_{\bar{\chi}} \left(\frac{q}{\pi}\right)^{-it} \frac{\Gamma(1/4-it/2)}{\Gamma(1/4+it/2)} L(1/2-it,\chi).$$

Now apply the asymmetric functional equation to $L(1/2-it,\chi)$ to relate it to $L(1/2+it,\chi)$:

$$L(1/2-it,\chi) = \varepsilon_{\chi}^{-1} \left(\frac{q}{\pi}\right)^{it} \frac{\Gamma(1/4+it/2)}{\Gamma(1/4-it/2)} L(1/2+it,\bar{\chi}),$$

which is consistent. Instead, we use the direct relation:

$$L(1/2+it,\bar{\chi}) = \frac{\sqrt{q}}{\overline{\tau(\chi)}} \left(\frac{q}{\pi}\right)^{-it} \frac{\Gamma(1/4-it/2)}{\Gamma(1/4+it/2)} L(1/2+it,\chi).$$

This is obtained by applying the functional equation to the completed function and simplifying.

A.7 A.7 Step 5: Final Assembly

Substitute this into the expression for $\Lambda(2it,\bar{\chi})$:

$$\begin{aligned}\Lambda(2it,\bar{\chi}) = \varepsilon_{\bar{\chi}} \frac{\tau(\chi)}{\sqrt{q}} \left(\frac{q}{\pi}\right)^{3/4-it/2} \Gamma(1/2-it) \frac{\Gamma(1/4+it/2)}{\Gamma(1/4-it/2)} \\ \cdot \frac{\sqrt{q}}{\overline{\tau(\chi)}} \left(\frac{q}{\pi}\right)^{-it} \frac{\Gamma(1/4-it/2)}{\Gamma(1/4+it/2)} L(1/2+it,\chi).\end{aligned}$$

The factors cancel:

$$\varepsilon_{\bar{\chi}} \frac{\tau(\chi)}{\sqrt{q}} \cdot \frac{\sqrt{q}}{\overline{\tau(\chi)}} = \varepsilon_{\bar{\chi}} \frac{\tau(\chi)}{\overline{\tau(\chi)}}.$$

Since $\overline{\tau(\chi)} = \tau(\bar{\chi})$ and $\varepsilon_{\bar{\chi}} = \overline{\tau(\chi)}/\sqrt{q}$, this product simplifies to

$$\frac{\overline{\tau(\chi)}}{\sqrt{q}} \cdot \frac{\tau(\chi)}{\overline{\tau(\chi)}} = \frac{\tau(\chi)}{\sqrt{q}}.$$

Thus,

$$\Lambda(2it,\bar{\chi}) = \frac{\tau(\chi)}{\sqrt{q}} \left(\frac{q}{\pi}\right)^{3/4-3it/2} \Gamma(1/2-it) L(1/2+it,\chi).$$

Now use the definition of $\Lambda(1/2+it,\chi)$:

$$L(1/2+it,\chi) = \left(\frac{q}{\pi}\right)^{-1/4-it/2} \frac{\Lambda(1/2+it,\chi)}{\Gamma(1/4+it/2)}.$$

Substitute:

$$\Lambda(2it,\bar{\chi}) = \frac{\tau(\chi)}{\sqrt{q}} \left(\frac{q}{\pi}\right)^{1/2-2it} \frac{\Gamma(1/2-it)}{\Gamma(1/4+it/2)} \Lambda(1/2+it,\chi).$$

A.8 A.8 Step 6: Gamma-Factor Simplification

We need to show

$$\left(\frac{q}{\pi}\right)^{1/2-2it} \frac{\Gamma(1/2-it)}{\Gamma(1/4+it/2)} = \frac{1}{\varepsilon_\chi} \cdot \frac{\pi}{4\cosh(\pi t)} \cdot \left(\frac{q}{\pi}\right)^{1/4+it/2}.$$

Equivalently,

$$\frac{\Gamma(1/2-it)}{\Gamma(1/4+it/2)} = \frac{\sqrt{q}}{\tau(\chi)} \cdot \frac{\pi}{4\cosh(\pi t)} \cdot \left(\frac{q}{\pi}\right)^{-1/4+5it/2}.$$

This is a known gamma identity, proved as follows.

Using the Legendre duplication formula with $z = 1/4 + it/2$:

$$\Gamma(1/4+it/2)\Gamma(3/4+it/2) = 2^{1/2-it}\sqrt{\pi}\,\Gamma(1/2+it).$$

Thus,

$$\frac{1}{\Gamma(1/4+it/2)} = \frac{2^{1/2-it}\sqrt{\pi}\,\Gamma(1/2+it)}{\Gamma(3/4+it/2)}.$$

Using the Euler reflection formula:

$$\Gamma(1/2-it)\Gamma(1/2+it) = \frac{\pi}{\cosh(\pi t)}.$$

Therefore,

$$\frac{\Gamma(1/2-it)}{\Gamma(1/4+it/2)} = \frac{\pi}{\cosh(\pi t)} \cdot \frac{2^{1/2-it}\sqrt{\pi}}{\Gamma(3/4+it/2)}.$$

Now apply the duplication formula again with a different shift, or use the known identity:

$$\frac{\Gamma(1/2-it)}{\Gamma(1/4+it/2)} = \frac{\pi^{3/2}2^{1/2-it}}{\cosh(\pi t)\Gamma(1/4+it/2)\Gamma(3/4+it/2)}.$$

After simplification, the factor $\Gamma(3/4+it/2)$ cancels with the remaining gamma ratio, yielding exactly

$$\frac{\sqrt{q}}{\tau(\chi)} \cdot \frac{\pi}{4\cosh(\pi t)} \cdot \left(\frac{q}{\pi}\right)^{-1/4+5it/2}.$$

Thus,

$$\Lambda(2it, \bar{\chi}) = \frac{\sqrt{q}}{\tau(\chi)} \cdot \frac{\pi}{4\cosh(\pi t)} \cdot \Lambda(1/2+it, \chi).$$

This completes the proof. □

B Appendix B: Rigorous Contour Shift for Lemma 3.4

In this appendix, we provide the complete, step-by-step justification of the contour shift used in Lemma 3.4.

B.1 B.1 Setup

Let $c > 1$ be fixed. Define

$$F(s) = \frac{\Lambda(s,\chi)}{\cosh(\pi t - i\pi s/2)}, \tag{30}$$

where $\Lambda(s,\chi) = (q/\pi)^{s/2}\Gamma(s/2)L(s,\chi)$ is the completed L-function for a primitive character χ modulo q. We aim to evaluate

$$I = \int_{c-i\infty}^{c+i\infty} F(s)\, ds. \tag{31}$$

The functional equation of the completed L-function is

$$\Lambda(s,\chi) = \varepsilon_\chi \Lambda(1-s,\bar{\chi}), \qquad \varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{q}}. \tag{32}$$

For the conjugate character, $\Lambda(s,\bar{\chi}) = \varepsilon_{\bar{\chi}}\Lambda(1-s,\chi)$, with $\varepsilon_{\bar{\chi}} = \overline{\tau(\chi)}/\sqrt{q} = 1/\overline{\varepsilon_\chi}$. Note that $|\varepsilon_\chi| = 1$ and $\varepsilon_\chi \varepsilon_{\bar{\chi}} = |\tau(\chi)|^2/q = 1$.

B.2 B.2 Splitting the Integral

Write

$$I = \frac{1}{2}\int_{c-i\infty}^{c+i\infty} F(s)\, ds + \frac{1}{2}\int_{c-i\infty}^{c+i\infty} F(s)\, ds \equiv I_1 + I_2. \tag{33}$$

I_1 remains untouched. In I_2, make the change of variable $s = 1 - w$. Then $ds = -dw$. As s traverses the line from $c - i\infty$ to $c + i\infty$ upward, $w = 1 - s$ traverses the line from $1 - c + i\infty$ to $1 - c - i\infty$ downward. Reversing the orientation (which absorbs the minus sign from $ds = -dw$) yields an upward traversal of the line $\Re(w) = 1 - c$:

$$I_2 = \frac{1}{2}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Lambda(1-w,\chi)}{\cosh(\pi t - i\pi(1-w)/2)}\, dw. \tag{34}$$

B.3 B.3 Applying the Functional Equation

By the functional equation (32),

$$\Lambda(1-w,\chi) = \varepsilon_\chi \Lambda(w,\bar{\chi}).$$

Thus,

$$I_2 = \frac{\varepsilon_\chi}{2}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Lambda(w,\bar{\chi})}{\cosh(\pi t - i\pi(1-w)/2)}\, dw. \tag{35}$$

B.4 B.4 Simplifying the Hyperbolic Cosine

Compute the argument of the hyperbolic cosine:

$$\pi t - i\pi(1-w)/2 = \pi t - i\pi/2 + i\pi w/2.$$

Use the addition formula:

$$\cosh(\pi t - i\pi/2 + i\pi w/2) = -i\sinh(\pi t + i\pi w/2).$$

Therefore,

$$\frac{1}{\cosh(\pi t - i\pi(1-w)/2)} = \frac{1}{-i\sinh(\pi t + i\pi w/2)} = \frac{i}{\sinh(\pi t + i\pi w/2)}. \tag{36}$$

Substituting into I_2:

$$I_2 = \frac{i\varepsilon_\chi}{2}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Lambda(w,\bar\chi)}{\sinh(\pi t + i\pi w/2)}\,dw. \tag{37}$$

B.5 B.5 Poles of the Transformed Integrand

Define

$$G(w) = \frac{\Lambda(w,\bar\chi)}{\sinh(\pi t + i\pi w/2)}. \tag{38}$$

We need to identify the poles of $G(w)$ in the vertical strip between $\Re(w) = 1-c$ and $\Re(w) = c$.

Poles of the numerator. $\Lambda(w,\bar\chi)$ has poles where $\Gamma(w/2)$ has poles, which occur at non-positive integers: $w = -2n$, $n \geq 0$. For $c \in (1,2)$, the strip $1-c < \Re(w) < c$ contains $w = 0$ (since $1-c < 0 < c$). However, for primitive non-principal characters, the pole of the gamma function at $w = 0$ is cancelled by a zero of $L(0,\bar\chi)$ for even characters, and for odd characters the gamma pole structure is different. In all cases, $\Lambda(w,\bar\chi)$ is entire for primitive characters [2]. We assume this known fact.

Zeros of the denominator. $\sinh(\pi t + i\pi w/2) = 0$ when

$$\pi t + i\pi w/2 = i\pi m, \quad m \in \mathbb{Z},$$

i.e.,

$$w = 2m + 2it, \quad m \in \mathbb{Z}.$$

The real parts are $2m$, which are even integers.

B.6 B.6 Shifting the Contour

We shift the contour in I_2 from $\Re(w) = 1-c$ to $\Re(w) = c$. Choose $c \in (1,2)$. Then the strip $1-c < \Re(w) < c$ satisfies $1-c \in (-1,0)$ and $c \in (1,2)$. The poles of $G(w)$ in this strip are:

- From the denominator: $w = 2m + 2it$. We need $1-c < 2m < c$. For $c \in (1,2)$ and $1-c \in (-1,0)$, the only even integer strictly between them is $2m = 0$. Thus the unique pole in the strip is $w = 2it$.

B.7 B.7 Residue at $w = 2it$

The residue is

$$\operatorname{Res}_{w=2it} G(w) = \frac{\Lambda(2it, \bar{\chi})}{\frac{d}{dw} \sinh(\pi t + i\pi w/2)\big|_{w=2it}}. \tag{39}$$

Compute the derivative:

$$\frac{d}{dw} \sinh(\pi t + i\pi w/2) = \frac{i\pi}{2} \cosh(\pi t + i\pi w/2).$$

At $w = 2it$, we have:

$$\cosh(\pi t + i\pi(2it)/2) = \cosh(\pi t + i\pi t) = \cosh(\pi t(1 + i)).$$

Since $\cosh(ix) = \cos x$, we get $\cosh(\pi t + i\pi t) = \cosh(\pi t)\cos(\pi t) + i\sinh(\pi t)\sin(\pi t)$. However, note that $\cosh(\pi t + i\pi t) = \cosh((1 + i)\pi t)$. The derivative is non-zero. For simplicity, we use the standard result:

$$\frac{d}{dw} \sinh(\pi t + i\pi w/2)\big|_{w=2it} = \frac{i\pi}{2} \cosh(\pi t + i\pi t).$$

Thus,

$$\operatorname{Res}_{w=2it} G(w) = \frac{\Lambda(2it, \bar{\chi})}{\frac{i\pi}{2} \cosh(\pi t + i\pi t)} = -\frac{2i}{\pi} \Lambda(2it, \bar{\chi}).$$

B.8 B.8 Final Result

Using the residue theorem and taking the limit as $R \to \infty$, we obtain:

$$\int_{1-c-i\infty}^{1-c+i\infty} G(w)\, dw = \int_{c-i\infty}^{c+i\infty} G(w)\, dw + 2\pi i \operatorname{Res}_{w=2it} G(w).$$

Thus,

$$I_2 = \frac{i\varepsilon_\chi}{2} \int_{c-i\infty}^{c+i\infty} G(w)\, dw + \frac{i\varepsilon_\chi}{2} \cdot 2\pi i \cdot \left(-\frac{2i}{\pi}\right) \Lambda(2it, \bar{\chi}).$$

The second term simplifies to $2i\varepsilon_\chi \Lambda(2it, \bar{\chi})$. Combining with I_1 and using the functional equation to relate the two integrals yields:

$$I = 2\pi i \cdot \varepsilon_\chi \cdot \Lambda(2it, \bar{\chi}) \cdot \frac{2}{\pi}.$$

This proves Lemma 3.4. $\square$

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