

Vanishing of Boundary Chern Classes under Derived Base Change on Toroidal Compactifications

Rahul Kundnani¹, Vinoth Marimuthu²,
Dr. Khursheed Alam³, Dr. Shri Kant⁴

^{1,2,3}Department of Mathematics & Data Science, Sharda University, Greater Noida, India

⁴University of Colorado Boulder, Boulder, CO 80309, United States

Abstract

We study the behavior of the top Chern class of the Hodge bundle on toroidal compactifications of moduli spaces of abelian varieties under derived base change. We distinguish throughout between ordinary operational restriction and refined/virtual Gysin pullback. The main result proves that the refined boundary cycle associated with the top Hodge Chern class is preserved under quasi-smooth derived base change. When the ordinary operational boundary restriction vanishes, the corresponding refined boundary cycle vanishes and hence so do all of its quasi-smooth derived boundary pullbacks. We further describe how quasi-smooth derived boundary tests factor through the refined boundary operator and record the resulting conditional consequence for tautological boundary kernels.

2020 Mathematics Subject Classification. 14A30, 14C17, 14C25, 14D23, 14K10, 14D06, 14F08.

Keywords. Toroidal compactifications; Hodge bundle; Chern classes; derived base change; derived fiber products; refined Gysin maps; operational Chow groups; tautological cycles; boundary vanishing; quasi-smooth morphisms.

1 Introduction

Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties and let

$$j : \mathcal{A}_g \hookrightarrow \mathcal{A}_g^{\text{tor}}$$

be a toroidal compactification. Let $\mathbb{E}$ be the extended Hodge bundle on $\mathcal{A}_g^{\text{tor}}$. The top Chern class

$$c_g(\mathbb{E}) := c_g(\mathbb{E})$$

is one of the central classes in the tautological intersection theory of $\mathcal{A}_g$ and its compactifications.

The purpose of this paper is to isolate a derived-functorial form of boundary vanishing. Suppose

$$f : \mathcal{Y} \longrightarrow \mathcal{A}_g^{\text{tor}}$$

is a quasi-smooth derived base change. We ask whether the vanishing of $c_g(\mathbb{E})$ along the boundary remains valid after pulling back to $\mathcal{Y}$. The main result proves that this is true when the boundary is formed as a derived fiber product.

If a test space meets the boundary non-transversely, the ordinary fiber product need not be Tor-independent and therefore does not itself retain the higher Tor data of the intersection. Classical refined intersection theory accounts for the resulting excess through the usual excess-intersection formalism. In the derived formulation used here, the same excess information is encoded intrinsically by the derived fiber product. Accordingly, our boundary-vanishing statement is formulated on the derived boundary square; this does not assert any failure of contravariant functoriality for ordinary operational Chow pullback.

2 Set-up and Conventions

Let

$$X = \mathcal{A}_g^{\text{tor}}$$

be a fixed toroidal compactification. We assume throughout that X is regular and that its boundary

$$D = X \setminus \mathcal{A}_g$$

is a relative strict normal crossings Cartier divisor; this is the standard toroidal compactification framework for moduli of principally polarized abelian varieties [2, Ch. IV, §1].

Let

$$i : D \hookrightarrow X$$

denote the boundary inclusion.

Definition 2.1. The extended Hodge bundle on X is denoted by $\mathbb{E}$; it is the extension of the Hodge bundle obtained from invariant differentials of the universal semi-abelian scheme over the toroidal compactification [2, Ch. IV, §1; Ch. V, §2]. Its top Chern class is

$$c_g(\mathbb{E}) = c_g(\mathbb{E}) \in \mathrm{CH}^g(X)_{\mathbb{Q}}.$$

Definition 2.2 (Operational and refined boundary classes). Let

$$c_g(\mathbb{E}) = c_g(\mathbb{E}) \in \mathrm{CH}^g(X)_{\mathbb{Q}}.$$

Its *operational boundary restriction* is

$$i^* c_g(\mathbb{E}) = i^* c_g(\mathbb{E}) \in \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

We say that $c_g(\mathbb{E})$ is *operationally boundary-vanishing* if

$$i^* c_g(\mathbb{E}) = 0 \quad \text{in } \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

Since X is regular and $i : D \hookrightarrow X$ is a regular immersion of codimension one, we also define the *refined boundary cycle*

$$\partial_D(c_g(\mathbb{E})) := i^!(c_g(\mathbb{E}) \cap [X]) \in \mathrm{CH}_{\dim X - g - 1}(D)_{\mathbb{Q}}.$$

We say that $c_g(\mathbb{E})$ is *refined-boundary-vanishing* if

$$\partial_D(c_g(\mathbb{E})) = 0.$$

By compatibility of operational classes with refined Gysin pullback,

$$i^!(c_g(\mathbb{E}) \cap [X]) = i^* c_g(\mathbb{E}) \cap [D].$$

Consequently,

$$i^* c_g(\mathbb{E}) = 0 \implies \partial_D(c_g(\mathbb{E})) = 0.$$

Remark 2.3. The use of rational Chow groups avoids torsion phenomena which are not relevant to the functorial mechanism studied here. The same formal argument may be transported to other oriented cohomology theories once Chern classes, refined pullbacks, and quasi-smooth base change are available.

Remark 2.4 (Operational pullback versus refined Gysin pullback). We use two distinct Chow-theoretic constructions and keep their notations separate throughout.

For a morphism $u : T \rightarrow Z$, ordinary contravariant pullback on operational Chow cohomology is denoted

$$u^* : \mathrm{CH}^r(Z)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^r(T)_{\mathbb{Q}}.$$

This pullback preserves codimension.

For a quasi-smooth morphism $u : T \rightarrow Z$ of virtual relative dimension d_u , the virtual Gysin pullback on Chow homology is denoted

$$u^! : \mathrm{CH}_k(Z)_{\mathbb{Q}} \longrightarrow \mathrm{CH}_{k+d_u}(T)_{\mathbb{Q}}.$$

For a classical regular immersion this agrees with the usual refined Gysin pullback. In particular, if u is a regular immersion of codimension c , then $d_u = -c$.

Operational classes act on cycle classes by cap product. Thus, for $\alpha \in \mathrm{CH}^r(X)_{\mathbb{Q}}$, the expression to which refined Gysin pullback is applied is

$$\alpha \cap [X] \in \mathrm{CH}_{\dim X - r}(X)_{\mathbb{Q}},$$

not the operational class α itself.

Remark 2.5 (Refined and virtual Gysin conventions). Refined pullbacks along classical regular immersions are used in the sense of Fulton's refined Gysin morphisms [1, Thm. 6.2, pp. 98–99; Thm. 6.5, p. 108].

For quasi-smooth morphisms of derived Artin stacks, virtual pullbacks are taken in the derived intersection-theoretic framework of [7, Sec. 3, especially Thms. 3.12–3.13]. When a classical Deligne–Mumford type morphism is presented together with a compatible perfect relative obstruction theory, the corresponding virtual pullback may equivalently be treated in Manolache's formalism [6, Def. 3.7, Rem. 3.9, Prop. 3.11, Thm. 4.1(i)].

Thus, in a derived Cartesian square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f_D} & D \\ i_{\mathcal{Y}} \downarrow & & \downarrow i \\ \mathcal{Y} & \xrightarrow{f} & X, \end{array}$$

with f quasi-smooth and i a Cartier divisor, both f and its derived base change f_D admit the virtual pullbacks used below, while i and $i_{\mathcal{Y}}$ carry the corresponding refined/virtual Gysin pullbacks. The base-change identity used throughout is

$$i_{\mathcal{Y}}^! \circ f^! = f_D^! \circ i^!.$$

This is an identity on Chow homology. No identity involving $i_{\mathcal{Y}}^*$ is inferred from this formula.

Proposition 2.6 (Stability along boundary strata). *Assume that the boundary is a relative strict normal crossings Cartier divisor on the regular space X , with irreducible Cartier components*

$$D = \bigcup_{a \in A} D_a.$$

For every non-empty finite subset $I \subset A$, put

$$D_I := \bigcap_{a \in I} D_a$$

with its induced scheme structure, and assume, as part of the strict normal crossings hypothesis, that every non-empty D_I is regular and that

$$i_I : D_I \hookrightarrow X$$

is an iterated regular immersion obtained by successive Cartier restrictions. If

$$i_a^* c_g(\mathbb{E}) = 0 \quad \text{in } \mathrm{CH}^g(D_a)_{\mathbb{Q}}$$

for every irreducible boundary component D_a , then

$$i_I^* c_g(\mathbb{E}) = 0 \quad \text{in } \mathrm{CH}^g(D_I)_{\mathbb{Q}}$$

for every non-empty stratum D_I .

Proof. Since D is a relative strict normal crossings Cartier divisor on the regular space X , each non-empty stratum D_I is regular and the closed immersion $i_I : D_I \hookrightarrow X$ is an iterated regular immersion obtained by successively restricting along Cartier components. Fix $I \neq \emptyset$ and choose $a \in I$. Then $D_I \hookrightarrow D_a$ is again an iterated Cartier, hence regular, immersion, and i_I factors as

$$D_I \xrightarrow{j_{I,a}} D_a \xrightarrow{i_a} X.$$

By contravariant functoriality of operational Chow pullback for the factorization $i_I = i_a \circ j_{I,a}$, one has

$$i_I^* c_g(\mathbb{E}) = j_{I,a}^* (i_a^* c_g(\mathbb{E})).$$

By hypothesis $i_a^* c_g(\mathbb{E}) = 0$ in $\mathrm{CH}^g(D_a)_{\mathbb{Q}}$. Therefore

$$i_I^* c_g(\mathbb{E}) = j_{I,a}^* (0) = 0$$

in $\mathrm{CH}^g(D_I)_{\mathbb{Q}}$. This proves the stability of boundary vanishing along every iterated boundary stratum. $\square$

3 Derived Boundary Squares

Let

$$f : \mathcal{Y} \longrightarrow X$$

be a morphism from a derived scheme or derived stack. The derived boundary of $\mathcal{Y}$ is defined by

$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D.$$

We write

$$i_{\mathcal{Y}} : \mathcal{D} \hookrightarrow \mathcal{Y}$$

for the induced boundary morphism and

$$f_D : \mathcal{D} \longrightarrow D$$

for the natural projection.

Definition 3.1 (Quasi-smooth morphisms). Let $f : \mathcal{Y} \rightarrow X$ be a morphism of derived schemes or derived Artin stacks. We say that f is *quasi-smooth* if f is locally of finite presentation and its relative cotangent complex

$$L_{\mathcal{Y}/X}$$

is perfect of Tor-amplitude contained in $[-1, 0]$.

On every connected component on which the virtual rank of $L_{\mathcal{Y}/X}$ is constant, we write

$$d_f := \mathrm{rk}(L_{\mathcal{Y}/X})$$

for the virtual relative dimension of f .

For a classical morphism of schemes, this condition recovers the usual local-complete-intersection condition.

Lemma 3.2 (Derived boundary base change and refined Gysin compatibility). *Let*

$$i : D \hookrightarrow X$$

be the boundary Cartier divisor, and let

$$f : \mathcal{Y} \rightarrow X$$

be quasi-smooth of virtual relative dimension d_f . Put

$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D.$$

Then

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f_D} & D \\ i_{\mathcal{Y}} \downarrow & & \downarrow i \\ \mathcal{Y} & \xrightarrow{f} & X \end{array}$$

is Cartesian in the derived sense. Moreover, $i_{\mathcal{Y}} : \mathcal{D} \rightarrow \mathcal{Y}$ is quasi-smooth of virtual codimension one, and $f_D : \mathcal{D} \rightarrow D$ is quasi-smooth of virtual relative dimension d_f .

For every cycle class

$$\xi \in \mathrm{CH}_k(X)_{\mathbb{Q}},$$

one has

$$i_{\mathcal{Y}}^!(f^!\xi) = f_D^!(i^!\xi) \quad \text{in } \mathrm{CH}_{k+d_f-1}(\mathcal{D})_{\mathbb{Q}}.$$

Proof. By definition,

$$\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D,$$

so the displayed square is Cartesian in the derived sense [9, Vol. II, Ch. 1, Sec. 4; Preface, Sec. 2.6]. No Tor-independence hypothesis is imposed.

Since $i : D \hookrightarrow X$ is a Cartier divisor, it is a regular immersion of codimension one and therefore quasi-smooth of virtual codimension one. Quasi-smoothness is preserved by derived base change, so

$$i_{\mathcal{Y}} : \mathcal{D} \hookrightarrow \mathcal{Y}$$

is quasi-smooth of virtual codimension one. Likewise, since f is quasi-smooth, its derived base change

$$f_D : \mathcal{D} \rightarrow D$$

is quasi-smooth with the same virtual relative dimension d_f .

The corresponding refined/virtual pullbacks are maps on Chow homology:

$$f^! : \mathrm{CH}_k(X)_{\mathbb{Q}} \rightarrow \mathrm{CH}_{k+d_f}(\mathcal{Y})_{\mathbb{Q}},$$

$$i_{\mathcal{Y}}^! : \mathrm{CH}_{k+d_f}(\mathcal{Y})_{\mathbb{Q}} \rightarrow \mathrm{CH}_{k+d_f-1}(\mathcal{D})_{\mathbb{Q}},$$

$$i^! : \mathrm{CH}_k(X)_{\mathbb{Q}} \rightarrow \mathrm{CH}_{k-1}(D)_{\mathbb{Q}},$$

and

$$f_D^! : \mathrm{CH}_{k-1}(D)_{\mathbb{Q}} \rightarrow \mathrm{CH}_{k+d_f-1}(\mathcal{D})_{\mathbb{Q}}.$$

By virtual pullback base-change compatibility for quasi-smooth morphisms in the derived Cartesian square [7, Sec. 3, especially Thms. 3.12–3.13], one has

$$i_{\mathcal{Y}}^! \circ f^! = f_D^! \circ i^!.$$

Therefore, for every $\xi \in \mathrm{CH}_k(X)_{\mathbb{Q}}$,

$$i_{\mathcal{Y}}^!(f^!\xi) = f_D^!(i^!\xi),$$

as required. $\square$

Lemma 3.3 (Derived pullback of the Hodge bundle and Chern-class functoriality). *Let $f : \mathcal{Y} \rightarrow X$ be any morphism for which derived pullback is defined, and let $\mathbb{E}$ be locally free of rank g on X . Then*

$$Lf^*\mathbb{E} \simeq f^*\mathbb{E},$$

so $Lf^\mathbb{E}$ is represented by a locally free sheaf of rank g and has Tor-amplitude 0. Moreover,*

$$c_g(Lf^*\mathbb{E}) = f^*c_g(\mathbb{E})$$

in operational Chow theory.

If, in addition, f is quasi-smooth of virtual relative dimension d_f , and if

$$[\mathcal{Y}]^{\mathrm{vir}} := f^![X],$$

then

$$f^!(c_g(\mathbb{E}) \cap [X]) = c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\mathrm{vir}}.$$

Proof. Since $\mathbb{E}$ is locally free on X , it is perfect of Tor-amplitude 0. Derived pullback preserves perfect complexes; see the standard perfect-module formalism in homotopical algebraic geometry [5, §1.3.7]. More specifically, local freeness gives

$$Lf^*\mathbb{E} \simeq f^*\mathbb{E},$$

so no higher Tor terms occur. This conclusion depends only on the local freeness of $\mathbb{E}$ and does not require f to be quasi-smooth.

For vector bundles, Chern classes satisfy pullback functoriality [1, Thm. 3.2(d), pp. 50–51]. Applied to the locally free representative $f^*\mathbb{E}$, this gives

$$c(Lf^*\mathbb{E}) = f^*c(\mathbb{E})$$

in operational Chow theory. Taking the degree g component yields

$$c_g(Lf^*\mathbb{E}) = f^*c_g(\mathbb{E}).$$

Now assume additionally that f is quasi-smooth. Compatibility of the associated virtual pullback with the action of operational classes gives

$$f^!(c_g(\mathbb{E}) \cap [X]) = f^*c_g(\mathbb{E}) \cap f^![X].$$

Writing

$$[\mathcal{Y}]^{\text{vir}} := f^![X]$$

and using

$$c_g(Lf^*\mathbb{E}) = f^*c_g(\mathbb{E}),$$

we obtain

$$f^!(c_g(\mathbb{E}) \cap [X]) = c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}.$$

This is the cycle-theoretic identity used in [Theorem 4.1](#). □

Remark 3.4 (Geometric input from the toroidal compactification). The proof of the main theorem is formally short, but its geometric and intersection-theoretic inputs must be distinguished. The toroidal compactification enters through three geometric inputs. First, the boundary $i : D \hookrightarrow X$ is a relative strict normal crossings Cartier divisor. Consequently, for any derived base change $f : \mathcal{Y} \rightarrow X$, the induced morphism

$$i_{\mathcal{Y}} : \mathcal{Y} \times_X^{\mathbf{L}} D \longrightarrow \mathcal{Y}$$

is the derived base change of the Cartier immersion i , and hence is quasi-smooth of virtual codimension one. This conclusion comes from the Cartier nature of $D \hookrightarrow X$, independently of whether f is quasi-smooth. The quasi-smoothness assumption on f is used separately to obtain the virtual pullbacks associated with f and f_D . Second, the extended Hodge bundle is a genuine locally free extension across the toroidal boundary, arising from the invariant differentials of the universal semi-abelian extension in toroidal compactification theory [2, Ch. IV, §1; Ch. V, §2]. Since $\mathbb{E}$ is locally free, Lemma 3.3 gives, for every morphism $f : \mathcal{Y} \rightarrow X$ for which derived pullback is defined,

$$Lf^*\mathbb{E} \simeq f^*\mathbb{E} \quad \text{and} \quad c_g(Lf^*\mathbb{E}) = f^*c_g(\mathbb{E})$$

in operational Chow theory. Thus the Chern-class pullback functoriality used in the argument is the algebraic functoriality of Chern classes of vector bundles. Third, the relevant refined boundary input is

$$\partial_D(c_g(\mathbb{E})) = i^!(c_g(\mathbb{E}) \cap [X]) = 0.$$

This condition is implied by the stronger operational restriction condition

$$i^*c_g(\mathbb{E}) = 0 \quad \text{in } \text{CH}^g(D)_{\mathbb{Q}},$$

because

$$i^!(c_g(\mathbb{E}) \cap [X]) = i^*c_g(\mathbb{E}) \cap [D].$$

The derived argument transports the refined cycle-theoretic boundary vanishing to non-transversal lci test families while retaining the derived excess-intersection data in

$$\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D.$$

4 Main Theorem

Theorem 4.1 (Derived preservation of refined boundary vanishing). *Let $X = \mathcal{A}_g^{\text{tor}}$ be regular of dimension n , with boundary Cartier divisor*

$$i : D \hookrightarrow X,$$

and let $\mathbb{E}$ denote the extended Hodge bundle on X . Assume that the refined boundary cycle of the top Hodge class vanishes:

$$i^!(c_g(\mathbb{E}) \cap [X]) = 0 \quad \text{in } \text{CH}_{n-g-1}(D)_{\mathbb{Q}}.$$

In particular, this hypothesis is satisfied whenever

$$i^*c_g(\mathbb{E}) = 0 \quad \text{in } \text{CH}^g(D)_{\mathbb{Q}}.$$

Let

$$f : \mathcal{Y} \longrightarrow X$$

be quasi-smooth of virtual relative dimension d_f , and put

$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D.$$

Write

$$[\mathcal{Y}]^{\text{vir}} := f^![X].$$

Then the pulled-back top Hodge class has vanishing refined boundary cycle:

$$i_{\mathcal{Y}}^!(c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}) = 0$$

in

$$\text{CH}_{n+d_f-g-1}(\mathcal{D})_{\mathbb{Q}}.$$

More precisely,

$$i_{\mathcal{Y}}^!(c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}) = f_D^!(i^!(c_g(\mathbb{E}) \cap [X])).$$

Proof. By [Lemma 3.3](#),

$$f^!(c_g(\mathbb{E}) \cap [X]) = c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}.$$

Applying the refined Gysin pullback

$$i_{\mathcal{Y}}^! : \text{CH}_{n+d_f-g}(\mathcal{Y})_{\mathbb{Q}} \longrightarrow \text{CH}_{n+d_f-g-1}(\mathcal{D})_{\mathbb{Q}}$$

gives

$$i_{\mathcal{Y}}^!(c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}) = i_{\mathcal{Y}}^!f^!(c_g(\mathbb{E}) \cap [X]).$$

By [Lemma 3.2](#),

$$i_{\mathcal{Y}}^! \circ f^! = f_D^! \circ i^!.$$

Hence

$$i_{\mathcal{Y}}^!(c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}) = f_D^!(i^!(c_g(\mathbb{E}) \cap [X])).$$

The assumed refined boundary vanishing gives

$$i^!(c_g(\mathbb{E}) \cap [X]) = 0,$$

so the right-hand side is $f_D^!(0) = 0$. This proves the assertion.

If instead one assumes the stronger operational condition

$$i^*c_g(\mathbb{E}) = 0,$$

then

$$i^!(c_g(\mathbb{E}) \cap [X]) = i^*c_g(\mathbb{E}) \cap [D] = 0,$$

and the same conclusion follows. $\square$

Theorem 4.2 (Universal refined boundary-vanishing criterion). *Let $n = \dim X$. The following conditions are equivalent:*

1.

$$i^!(c_g(\mathbb{E}) \cap [X]) = 0 \quad \text{in } \text{CH}_{n-g-1}(D)_{\mathbb{Q}}.$$

2. For every quasi-smooth derived test space

$$f : \mathcal{Y} \longrightarrow X$$

of virtual relative dimension d_f , with

$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D,$$

one has

$$i_{\mathcal{Y}}^! \left(c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}} \right) = 0$$

in

$$\text{CH}_{n+d_f-g-1}(\mathcal{D})_{\mathbb{Q}}.$$

3. For every regular complete-intersection immersion

$$f : S \hookrightarrow X$$

of codimension c , with

$$S_D := S \times_X^{\mathbf{L}} D,$$

one has

$$i_S^! \left(c_g(\mathbb{E}|_S) \cap [S] \right) = 0 \quad \text{in } \text{CH}_{n-c-g-1}(S_D)_{\mathbb{Q}}.$$

Proof. The implication (1) $\Rightarrow$ (2) is [Theorem 4.1](#).

For (2) $\Rightarrow$ (3), let $f : S \hookrightarrow X$ be a regular complete-intersection immersion of codimension c . Then

$$f^! [X] = [S]$$

and, since $\mathbb{E}$ is locally free,

$$Lf^*\mathbb{E} \simeq \mathbb{E}|_S.$$

Condition (2) therefore gives

$$i_S^! \left(c_g(\mathbb{E}|_S) \cap [S] \right) = 0.$$

Finally, for (3) $\Rightarrow$ (1), take the identity immersion

$$S = X.$$

Then

$$S_D = X \times_X^{\mathbf{L}} D = D, \quad i_S = i, \quad [S] = [X],$$

so condition (3) becomes exactly

$$i^! (c_g(\mathbb{E}) \cap [X]) = 0.$$

□

Remark 4.3 (Specialization-compatible deformation of boundary vanishing). Let

$$\pi : \mathcal{Z} \rightarrow B$$

be a flat derived family over a regular base, and let

$$F : \mathcal{Z} \rightarrow X$$

be a morphism whose geometric fibres

$$F_b : \mathcal{Z}_b \rightarrow X$$

are quasi-smooth.

For each geometric fibre $F_b : \mathcal{Z}_b \rightarrow X$, let d_b be its virtual relative dimension and write

$$[\mathcal{Z}_b]^{\text{vir}} := F_b^! [X].$$

Put

$$\mathcal{D}_b := \mathcal{Z}_b \times_X^{\mathbf{L}} D.$$

Then

$$i_{\mathcal{Z}_b}^! \left(c_g(LF_b^*\mathbb{E}) \cap [\mathcal{Z}_b]^{\text{vir}} \right) = F_{D,b}^! \left(i^! (c_g(\mathbb{E}) \cap [X]) \right).$$

Therefore, if

$$i^! (c_g(\mathbb{E}) \cap [X]) = 0,$$

then

$$i_{\mathcal{Z}_b}^! \left(c_g(LF_b^* \mathbb{E}) \cap [\mathcal{Z}_b]^{\text{vir}} \right) = 0$$

for every geometric fibre b .

Without a specified specialization isomorphism, locally constant Chow class system, or parallel-transport mechanism for the relevant Chow groups, vanishing on a single fibre b_0 alone does not imply vanishing on every fibre. Thus the deformation statement used here is the fibrewise consequence of the global refined boundary-vanishing input, not an independent propagation theorem from one fibre to all fibres.

Example 4.4 (The boundary cusp for $g = 1$). Let

$$X = \overline{\mathcal{M}}_{1,1}$$

be the compactified moduli stack of elliptic curves, and let

$$i : D \hookrightarrow X$$

be its boundary divisor. Thus D is the cusp, parametrizing stable nodal cubic curves. In this case the extended Hodge bundle $\mathbb{E}$ is a line bundle, usually denoted by λ , and one has the standard rational relation

$$c_1(\mathbb{E}) = \lambda = \frac{1}{12}[D] \quad \text{in } \text{CH}^1(X)_{\mathbb{Q}}.$$

Now let

$$f : \mathcal{Y} \longrightarrow X$$

be any quasi-smooth derived test space and form the derived boundary

$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D.$$

The operational restriction condition

$$i^* c_1(\mathbb{E}) = 0 \quad \text{in } \text{CH}^1(D)_{\mathbb{Q}}$$

implies

$$i^! (c_1(\mathbb{E}) \cap [X]) = 0.$$

Hence [Theorem 4.1](#) gives

$$i_{\mathcal{Y}}^! \left(c_1(Lf^* \mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}} \right) = 0.$$

More precisely,

$$i_{\mathcal{Y}}^! \left(c_1(Lf^* \mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}} \right) = f_D^! \left(i^! (c_1(\mathbb{E}) \cap [X]) \right).$$

This is the refined base-change identity on the derived boundary.

This example illustrates the general mechanism of the paper in its simplest modular form: the refined boundary-vanishing identity is formulated on the derived boundary

$$\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D,$$

which retains the higher Tor data of a non-Tor-independent intersection. When one instead works with the classical boundary, the corresponding comparison is governed by the classical refined/excess-intersection formalism.

5 Consequences for Tautological Cycles

Let

$$R^*(X) \subseteq \text{CH}^*(X)_{\mathbb{Q}}$$

denote the tautological subring generated by Chern classes of the Hodge bundle.

For a collection $\mathcal{T}$ of quasi-smooth morphisms $f : \mathcal{Y} \rightarrow X$, define

$$K_{\mathcal{T}} := \{ \alpha \in R^*(X) : i_{\mathcal{Y}}^! f^! (\alpha \cap [X]) = 0 \text{ for every } f \in \mathcal{T} \}.$$

For a homogeneous class $\alpha \in R^r(X)$ and an quasi-smooth test f of virtual relative dimension d_f , the corresponding test class lies in

$$\text{CH}_{\dim X - r + d_f - 1}(\mathcal{Y} \times_X^{\mathbf{L}} D)_{\mathbb{Q}}.$$

Definition 5.1 (Boundary-separating quasi-smooth test systems). Define the refined boundary operator

$$\partial_D : R^*(X) \longrightarrow \mathrm{CH}_*(D)_{\mathbb{Q}}, \quad \partial_D(\alpha) := i^!(\alpha \cap [X]).$$

For a collection $\mathcal{T}$ of quasi-smooth morphisms $f : \mathcal{Y} \rightarrow X$, define the boundary-cycle test map

$$\rho_{\mathcal{T}} : \mathrm{im}(\partial_D) \longrightarrow \prod_{f \in \mathcal{T}} \mathrm{CH}_*(\mathcal{Y} \times_X^{\mathbb{L}} D)_{\mathbb{Q}}$$

by

$$\rho_{\mathcal{T}}(\gamma) := (f_D^!(\gamma))_{f \in \mathcal{T}}.$$

We call $\mathcal{T}$ boundary-separating if $\rho_{\mathcal{T}}$ is injective.

Remark 5.2 (Principal refined boundary-kernel hypothesis). The hypothesis

$$\ker(\partial_D) = (c_g(\mathbb{E}))$$

is an independent tautological input. It is not deduced from derived base change or from the boundary-separating property of a test system.

Proposition 5.3 (Factorization of derived boundary tests). *Let $\mathcal{T}$ be a collection of quasi-smooth morphisms $f : \mathcal{Y} \rightarrow X$. For every $\alpha \in R^*(X)$ and every $f \in \mathcal{T}$,*

$$i_{\mathcal{Y}}^! f^!(\alpha \cap [X]) = f_D^!(\partial_D(\alpha)).$$

Consequently, the combined test map

$$\Phi_{\mathcal{T}} : R^*(X) \longrightarrow \prod_{f \in \mathcal{T}} \mathrm{CH}_*(\mathcal{Y} \times_X^{\mathbb{L}} D)_{\mathbb{Q}},$$

$$\Phi_{\mathcal{T}}(\alpha) := (i_{\mathcal{Y}}^! f^!(\alpha \cap [X]))_{f \in \mathcal{T}},$$

factors as

$$R^*(X) \xrightarrow{\partial_D} \mathrm{im}(\partial_D) \xrightarrow{\rho_{\mathcal{T}}} \prod_{f \in \mathcal{T}} \mathrm{CH}_*(\mathcal{Y} \times_X^{\mathbb{L}} D)_{\mathbb{Q}}.$$

If $\mathcal{T}$ is boundary-separating, then

$$\ker(\Phi_{\mathcal{T}}) = \ker(\partial_D).$$

Proof. For every $f \in \mathcal{T}$, [Lemma 3.2](#) gives

$$i_{\mathcal{Y}}^! f^!(\alpha \cap [X]) = f_D^! i^!(\alpha \cap [X]) = f_D^!(\partial_D(\alpha)).$$

Hence

$$\Phi_{\mathcal{T}} = \rho_{\mathcal{T}} \circ \partial_D.$$

It follows immediately that

$$\ker(\partial_D) \subseteq \ker(\Phi_{\mathcal{T}}).$$

If $\mathcal{T}$ is boundary-separating and $\alpha \in \ker(\Phi_{\mathcal{T}})$, then

$$\rho_{\mathcal{T}}(\partial_D(\alpha)) = 0.$$

Since $\partial_D(\alpha) \in \mathrm{im}(\partial_D)$ and $\rho_{\mathcal{T}}$ is injective on this image,

$$\partial_D(\alpha) = 0.$$

Thus

$$\ker(\Phi_{\mathcal{T}}) \subseteq \ker(\partial_D),$$

and therefore

$$\ker(\Phi_{\mathcal{T}}) = \ker(\partial_D).$$

□

Corollary 5.4 (Conditional tautological boundary detection). *Assume independently that*

$$\ker(\partial_D) = (c_g(\mathbb{E})).$$

If $\mathcal{T}$ is boundary-separating in the sense of Definition 5.1, then

$$K_{\mathcal{T}} = (c_g(\mathbb{E})).$$

Equivalently, for every $\alpha \in R^(X)$,*

$$\alpha \in (c_g(\mathbb{E})) \iff i_{\mathcal{Y}}^! f^!(\alpha \cap [X]) = 0 \text{ for every } f \in \mathcal{T}.$$

Proof. By definition,

$$K_{\mathcal{T}} = \ker(\Phi_{\mathcal{T}}).$$

By Proposition 5.3, boundary separation gives

$$\ker(\Phi_{\mathcal{T}}) = \ker(\partial_D).$$

The independent principal-kernel hypothesis therefore yields

$$K_{\mathcal{T}} = \ker(\partial_D) = (c_g(\mathbb{E})).$$

□

Remark 5.5 (Refined-cycle interpretation of derived boundary vanishing). Under the assumptions of Theorem 4.1, the pulled-back top Hodge class satisfies

$$i_{\mathcal{Y}}^! (c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}) = 0 \text{ in } \text{CH}_{\dim X + d_f - g - 1}(\mathcal{D})_{\mathbb{Q}}.$$

Equivalently, the virtual cycle

$$c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}$$

has trivial refined boundary cycle along

$$i_{\mathcal{Y}} : \mathcal{D} \rightarrow \mathcal{Y}.$$

Proposition 5.6 (Refined boundary contribution of a decomposed tautological class). *Suppose that a homogeneous tautological class $\alpha \in R^g(X)$ admits a decomposition*

$$\alpha = c_g(\mathbb{E}) + \beta.$$

Assume moreover that the refined boundary cycle of the top Hodge class vanishes:

$$\partial_D(c_g(\mathbb{E})) = i^!(c_g(\mathbb{E}) \cap [X]) = 0.$$

Let

$$f : \mathcal{Y} \longrightarrow X$$

be any quasi-smooth derived base change of virtual relative dimension d_f , and put

$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D, \quad [\mathcal{Y}]^{\text{vir}} := f^![X].$$

Then

$$i_{\mathcal{Y}}^! f^!(\alpha \cap [X]) = i_{\mathcal{Y}}^! f^!(\beta \cap [X]).$$

in

$$\text{CH}_{\dim X + d_f - g - 1}(\mathcal{D})_{\mathbb{Q}}.$$

In particular, if

$$i_{\mathcal{Y}}^! f^!(\beta \cap [X]) = 0,$$

then

$$i_{\mathcal{Y}}^! f^!(\alpha \cap [X]) = 0.$$

Proof. By the decomposition

$$\alpha = c_g(\mathbb{E}) + \beta$$

and the linearity of the virtual pullback $f^!$, we have

$$f^!(\alpha \cap [X]) = f^!(c_g(\mathbb{E}) \cap [X]) + f^!(\beta \cap [X]).$$

Applying the refined boundary pullback $i_{\mathcal{Y}}^!$ gives

$$i_{\mathcal{Y}}^! f^!(\alpha \cap [X]) = i_{\mathcal{Y}}^! f^!(c_g(\mathbb{E}) \cap [X]) + i_{\mathcal{Y}}^! f^!(\beta \cap [X]).$$

By the assumed refined boundary vanishing and Theorem 4.1,

$$i_{\mathcal{Y}}^! f^!(c_g(\mathbb{E}) \cap [X]) = 0.$$

Therefore

$$i_{\mathcal{Y}}^! f^!(\alpha \cap [X]) = i_{\mathcal{Y}}^! f^!(\beta \cap [X]).$$

The final assertion follows immediately. □

6 Non-Transversal Intersections

The derived formulation becomes most visible when the test space does not meet the boundary transversely.

Example 6.1 (A complete-intersection test family meeting the boundary non-transversely). Let

$$S \hookrightarrow X$$

be a regular closed immersion cutting out a complete-intersection test family inside $X = \mathcal{A}_g^{\text{tor}}$. Thus the morphism

$$f : S \longrightarrow X$$

is quasi-smooth. Let $i : D \hookrightarrow X$ be the boundary divisor and form the derived boundary of S :

$$S_D := S \times_X^{\mathbf{L}} D.$$

If S meets D transversely, then the pair (S, D) is Tor-independent over X . In this case the derived fiber product has no higher homology and coincides with the ordinary fiber product:

$$S \times_X^{\mathbf{L}} D = S \times_X D.$$

The refined base-change identity of Lemma 3.2 then reduces to the usual classical compatibility

$$i_S^! f^! (\gamma \cap [X]) = f_D^! i^! (\gamma \cap [X]), \quad \gamma \in \text{CH}^*(X)_{\mathbb{Q}}.$$

In particular, applying this to $\gamma = c_g(\mathbb{E})$, Theorem 4.1 gives

$$i_S^! (c_g(\mathbb{E}|_S) \cap [S]) = 0,$$

provided

$$i^! (c_g(\mathbb{E}) \cap [X]) = 0.$$

The derived enhancement becomes nontrivial when S meets D non-transversely. Locally on X , suppose that the boundary divisor is given by one equation

$$t = 0,$$

and that S is cut out by a regular sequence

$$a_1, \dots, a_r.$$

The ordinary boundary intersection is then described by the quotient

$$\mathcal{O}_X / (a_1, \dots, a_r, t).$$

If t fails to remain a non-zero-divisor after restriction to $\mathcal{O}_S = \mathcal{O}_X / (a_1, \dots, a_r)$, then the intersection is not Tor-independent. The classical fiber product $S \times_X D$ has structure sheaf

$$\mathcal{O}_X / (a_1, \dots, a_r, t),$$

while the derived fiber product additionally retains the higher Tor data:

$$\mathcal{O}_{S_D} \simeq \mathcal{O}_S \otimes_X^{\mathbf{L}} \mathcal{O}_D.$$

Equivalently, S_D is represented locally by the two-term derived intersection complex

$$\left[\mathcal{O}_S \xrightarrow{t} \mathcal{O}_S \right],$$

whose degree -1 homology measures the failure of t to be a non-zero-divisor on $\mathcal{O}_S$. This higher homology records the excess-intersection data intrinsically in the derived structure; in the classical refined-intersection formalism the corresponding phenomenon is handled by the usual excess-intersection construction.

The derived boundary S_D is the target of the refined Gysin pullback, and Lemma 3.2 gives

$$i_S^! f^! (c_g(\mathbb{E}) \cap [X]) = f_D^! i^! (c_g(\mathbb{E}) \cap [X]).$$

Since

$$f^! (c_g(\mathbb{E}) \cap [X]) = c_g(Lf^* \mathbb{E}) \cap [S]$$

for the regular immersion $f : S \hookrightarrow X$, refined boundary vanishing on D implies

$$i_S^! (c_g(Lf^* \mathbb{E}) \cap [S]) = 0.$$

This example compares the classical refined and derived formulations. In the Tor-independent case the derived fiber product agrees with the classical fiber product, and the identity reduces to the usual classical refined pullback statement. In the non-Tor-independent case, classical refined intersection theory expresses the excess contribution through its excess-intersection formalism, whereas the derived fiber product encodes the corresponding higher Tor data directly in its derived structure. The vanishing identity used here is the refined/virtual identity on that derived boundary.

Remark 6.2 (Classical truncation and excess intersection). If one replaces the derived boundary $S \times_X^{\mathbf{L}} D$ by the classical fiber product $S \times_X D$ in a non-Tor-independent square, the corresponding classical refined-intersection formula requires the usual excess-intersection correction. The identity used in this paper is instead the virtual/refined identity on Chow homology

$$i_S^! \circ f^! = f_D^! \circ i^!$$

for the derived Cartesian square. No claim is made that ordinary operational pullback i_S^* fails to be contravariantly functorial.

Proposition 6.3 (Excess-intersection interpretation). *Let*

$$f : Y \longrightarrow X$$

be a quasi-smooth morphism of classical schemes (equivalently, an lci morphism in the classical setting). Form the derived boundary

$$\mathcal{D} := Y \times_X^{\mathbf{L}} D$$

and its classical truncation

$$D_{\text{cl}} := t_0(\mathcal{D}) = Y \times_X D.$$

Assume that the intersection of the classical schemes Y and D over X is not Tor-independent, equivalently that locally

$$\text{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D) \neq 0.$$

Then, because $D \hookrightarrow X$ is Cartier, $\mathcal{O}_D$ has Tor-dimension at most one over $\mathcal{O}_X$. Hence

$$\text{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D) = 0 \quad \text{for every } q > 1.$$

Since Y and D are classical, the homology sheaves of

$$\mathcal{O}_{\mathcal{D}} \simeq \mathcal{O}_Y \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_D$$

are the ordinary Tor sheaves

$$H^{-q}(\mathcal{O}_{\mathcal{D}}) \simeq \text{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D).$$

In particular,

$$H^0(\mathcal{O}_{\mathcal{D}}) = \mathcal{O}_{D_{\text{cl}}},$$

and therefore the virtual structure sheaf of the derived boundary satisfies

$$[\mathcal{O}_{\mathcal{D}}]^{\text{vir}} = [\mathcal{O}_{D_{\text{cl}}}] - [\text{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D)] \quad \text{in } K_0(D_{\text{cl}}).$$

Define

$$e_{\text{ex}}(Y, D/X) := -[\text{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D)] \in K_0(D_{\text{cl}}).$$

Then

$$[\mathcal{O}_{\mathcal{D}}]^{\text{vir}} = [\mathcal{O}_{D_{\text{cl}}}] + e_{\text{ex}}(Y, D/X).$$

For every homogeneous operational Chow class

$$\alpha \in \text{CH}^r(X)_{\mathbb{Q}},$$

the corresponding refined derived boundary cycle satisfies

$$i_Y^! \left(f^! (\alpha \cap [X]) \right) = f_D^! \left(i^! (\alpha \cap [X]) \right)$$

in

$$\text{CH}_{\dim X - r + d_f - 1}(\mathcal{D})_{\mathbb{Q}}.$$

Thus the possible nonzero ordinary Tor sheaf

$$\text{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D)$$

measures the discrepancy between the classical truncation $D_{\text{cl}} = Y \times_X D$ and the derived boundary $\mathcal{D} = Y \times_X^{\mathbf{L}} D$. Because $D \hookrightarrow X$ is Cartier, no higher Tor sheaves occur in degrees $q > 1$. The refined identity itself is taken on the derived boundary $\mathcal{D}$.

In particular, for

$$\alpha = c_g(\mathbb{E}),$$

one has

$$i_Y^! \left(c_g(f^* \mathbb{E}) \cap [Y]^{\text{vir}} \right) = f_D^! \left(i^! (c_g(\mathbb{E}) \cap [X]) \right).$$

Hence, if

$$i^! (c_g(\mathbb{E}) \cap [X]) = 0,$$

then

$$i_Y^! \left(c_g(f^* \mathbb{E}) \cap [Y]^{\text{vir}} \right) = 0.$$

Proof. Since Y , X , and D are classical schemes and

$$\mathcal{D} = Y \times_X^{\mathbf{L}} D,$$

the structure sheaf of the derived boundary is represented by

$$\mathcal{O}_{\mathcal{D}} \simeq \mathcal{O}_Y \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_D.$$

Moreover,

$$t_0(\mathcal{D}) = Y \times_X D = D_{\text{cl}}.$$

Since $D \hookrightarrow X$ is Cartier, locally D is cut out by a non-zero-divisor t , and $\mathcal{O}_D$ has the length-one locally free resolution

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{t} \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

After derived tensoring with the ordinary structure sheaf $\mathcal{O}_Y$, the derived boundary is therefore represented locally by the two-term complex

$$[\mathcal{O}_Y \xrightarrow{t} \mathcal{O}_Y].$$

Consequently,

$$H^{-q}(\mathcal{O}_{\mathcal{D}}) \simeq \text{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D),$$

so

$$\text{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D) = 0 \quad (q > 1),$$

while

$$H^0(\mathcal{O}_{\mathcal{D}}) = \text{Tor}_0^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D) = \mathcal{O}_{D_{\text{cl}}}.$$

Hence the alternating K -theory class of the derived structure sheaf is

$$[\mathcal{O}_{\mathcal{D}}]^{\text{vir}} = [\mathcal{O}_{D_{\text{cl}}}] - [\text{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D)]$$

in $K_0(D_{\text{cl}})$, which is the asserted formula for $e_{\text{ex}}(Y, D/X)$.

The refined identity

$$i_Y^! \left(f^! (\alpha \cap [X]) \right) = f_D^! \left(i^! (\alpha \cap [X]) \right)$$

is precisely the derived base-change compatibility of [Lemma 3.2](#). The equality is taken on the derived boundary

$$\mathcal{D} = Y \times_X^{\mathbf{L}} D.$$

Its classical truncation is the ordinary fiber product

$$t_0(\mathcal{D}) = D_{\text{cl}} = Y \times_X D,$$

and the nonzero Tor_1 -sheaf records the discrepancy between these two objects.

Applying this identity to

$$\alpha = c_g(\mathbb{E})$$

and using

$$f^! (c_g(\mathbb{E}) \cap [X]) = c_g(f^* \mathbb{E}) \cap [Y]^{\text{vir}},$$

gives

$$i_Y^! \left(c_g(f^* \mathbb{E}) \cap [Y]^{\text{vir}} \right) = f_D^! \left(i^! (c_g(\mathbb{E}) \cap [X]) \right).$$

If

$$i^! (c_g(\mathbb{E}) \cap [X]) = 0,$$

then the right-hand side is zero. Hence

$$i_Y^! \left(c_g(f^* \mathbb{E}) \cap [Y]^{\text{vir}} \right) = 0.$$

□

Example 6.4 (A derived complete-intersection test family computed by a Koszul complex). We now make the excess term in [Proposition 6.3](#) completely explicit in a local complete-intersection test family.

All schemes in this example are classical; the derived object enters only when the derived boundary fiber product is formed.

Let

$$X = \operatorname{Spec} R, \quad R = k[x, y, z],$$

and let the boundary divisor be

$$D = V(x) \hookrightarrow X.$$

Thus D is Cartier, with

$$\mathcal{O}_D = R/(x).$$

Let

$$S = V(xy, z) \hookrightarrow X.$$

Since xy, z is a regular sequence in R , the morphism of classical schemes

$$f : S \longrightarrow X$$

is a regular complete-intersection morphism, hence an lci and therefore quasi-smooth test morphism when regarded in derived geometry.

The ordinary boundary intersection is

$$S \times_X D = \operatorname{Spec} R/(xy, z, x) = \operatorname{Spec} k[y].$$

The classical intersection records the degree-zero intersection scheme, whereas its derived enhancement additionally records the higher Tor data. Indeed, the derived boundary is

$$S_D := S \times_X^{\mathbf{L}} D,$$

whose classical truncation is

$$t_0(S_D) = S \times_X D = \operatorname{Spec} k[y].$$

Its structure sheaf is computed by the derived tensor product

$$\mathcal{O}_{S_D} \simeq \mathcal{O}_S \otimes_R^{\mathbf{L}} \mathcal{O}_D.$$

Using the Koszul resolution of $\mathcal{O}_D = R/(x)$, this is represented by the two-term complex

$$\left[R/(xy, z) \xrightarrow{x} R/(xy, z) \right].$$

Therefore

$$H^0(\mathcal{O}_{S_D}) = R/(xy, z, x) = k[y],$$

while

$$H^{-1}(\mathcal{O}_{S_D}) = \ker(x : R/(xy, z) \rightarrow R/(xy, z)).$$

Since $xy = 0$ in $R/(xy, z)$, multiplication by x kills the class of y . Hence

$$H^{-1}(\mathcal{O}_{S_D}) \simeq (y)/(xy, z) \simeq k[y].$$

Equivalently,

$$\operatorname{Tor}_1^R(\mathcal{O}_S, \mathcal{O}_D) \simeq k[y] \neq 0.$$

Thus the virtual structure sheaf of the derived boundary is

$$[\mathcal{O}_{S_D}]^{\operatorname{vir}} = [\mathcal{O}_{S \times_X D}] - [\operatorname{Tor}_1^R(\mathcal{O}_S, \mathcal{O}_D)] \quad \text{in } K_0(S \times_X D).$$

In this example,

$$[\mathcal{O}_{S \times_X D}] = [k[y]], \quad [\operatorname{Tor}_1^R(\mathcal{O}_S, \mathcal{O}_D)] = [k[y]],$$

so the excess class is

$$e_{\operatorname{ex}}(S, D/X) = -[k[y]].$$

This explicitly illustrates [Proposition 6.3](#): the classical fiber product $S \times_X D$ has the displayed degree-zero structure sheaf, while the derived fiber product additionally retains the nonzero Tor_1 -term. Classical refined intersection theory accounts for the corresponding excess through its excess-intersection formalism. The refined Gysin identities used here are identities on Chow homology associated with the derived boundary square and are distinct from the contravariant functoriality of ordinary operational restriction maps.

Proposition 6.5 (Refined base-change compatibility for Hodge cycles). *Let $f : \mathcal{Y} \rightarrow X$ be quasi-smooth of virtual relative dimension d_f , and let*

$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D.$$

Then

$$i_{\mathcal{Y}}^! \left(c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}} \right) = f_D^! \left(i^! (c_g(\mathbb{E}) \cap [X]) \right)$$

in

$$\text{CH}_{\dim X + d_f - g - 1}(\mathcal{D})_{\mathbb{Q}}.$$

This identity requires no Tor-independence assumption between f and i .

Proof. By Lemma 3.3,

$$f^! (c_g(\mathbb{E}) \cap [X]) = c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}}.$$

By Lemma 3.2,

$$i_{\mathcal{Y}}^! \circ f^! = f_D^! \circ i^!.$$

Therefore

$$i_{\mathcal{Y}}^! \left(c_g(Lf^*\mathbb{E}) \cap [\mathcal{Y}]^{\text{vir}} \right) = f_D^! \left(i^! (c_g(\mathbb{E}) \cap [X]) \right),$$

as claimed. □

Proposition 6.6 (Restriction to regular complete-intersection test families). *Let*

$$f : S \hookrightarrow X$$

be a regular immersion of codimension c , and let

$$S_D := S \times_X^{\mathbf{L}} D.$$

Assume that

$$i^! (c_g(\mathbb{E}) \cap [X]) = 0.$$

Then

$$i_S^! \left(c_g(\mathbb{E}|_S) \cap [S] \right) = 0$$

in

$$\text{CH}_{\dim X - c - g - 1}(S_D)_{\mathbb{Q}}.$$

Proof. Since $f : S \hookrightarrow X$ is a regular immersion,

$$f^! [X] = [S].$$

Since $\mathbb{E}$ is locally free,

$$Lf^*\mathbb{E} \simeq \mathbb{E}|_S.$$

Hence

$$f^! (c_g(\mathbb{E}) \cap [X]) = c_g(\mathbb{E}|_S) \cap [S].$$

Applying Theorem 4.1 gives

$$i_S^! \left(c_g(\mathbb{E}|_S) \cap [S] \right) = 0.$$

□

Remark 6.7. The proposition provides the geometric interpretation of Theorem 4.1. In practice, regular immersions arise from complete-intersection subvarieties, modular test families, and tautological cycles inside toroidal compactifications. The statement shows that boundary vanishing of the top Hodge class is inherited by all such test families after passage to their derived boundaries.

When S meets the boundary divisor D Tor-independently, the derived boundary S_D agrees with the classical fiber product $S \times_X D$, and the proposition reduces to the corresponding classical refined statement. When the intersection is not Tor-independent, the derived boundary retains the higher Tor data and the displayed refined/virtual identity requires no separate excess term on the derived side. The analogous classical refined intersection statement is expressed using the usual excess-intersection formalism.

7 Role and Scope of the Hypotheses

The assumptions in [Theorem 4.1](#) play distinct roles and should not be interpreted as having the same logical status. The regularity of X provides the cycle-theoretic setting used for the fundamental class $[X]$. The Cartier hypothesis on $i : D \hookrightarrow X$ provides the codimension-one refined boundary pullback and, independently of the quasi-smoothness of f , implies that its derived base change

$$i_{\mathcal{Y}} : \mathcal{Y} \times_X^L D \longrightarrow \mathcal{Y}$$

is quasi-smooth of virtual codimension one. The quasi-smoothness hypothesis on f has a separate role: it places f and its derived base change f_D in the virtual-pullback framework in which

$$f^!, \quad f_D^!$$

are defined and the compatibility

$$i_{\mathcal{Y}}^! \circ f^! = f_D^! \circ i^!$$

is available. Thus the quasi-smoothness of f is the standard geometric hypothesis used in the derived virtual-pullback formulation of the theorem; it is not the source of the quasi-smoothness of $i_{\mathcal{Y}}$, which follows instead from the Cartier property of the boundary.

Example 7.1 (Role of the Cartier boundary hypothesis). The Cartier hypothesis on the boundary divisor is essential for the specific codimension-one refined Gysin mechanism used in [Theorem 4.1](#). To illustrate this, consider a hypothetical compactification X whose boundary

$$D = X \setminus \mathcal{A}_g$$

is not a Cartier divisor.

Locally, a Cartier boundary is cut out by a single non-zero-divisor $t \in \mathcal{O}_X$. This property allows one to view the inclusion

$$i : D \hookrightarrow X$$

as a regular immersion of codimension one. Consequently, the refined Gysin pullback

$$i^! : \mathrm{CH}_k(X)_{\mathbb{Q}} \longrightarrow \mathrm{CH}_{k-1}(D)_{\mathbb{Q}}$$

is defined in the classical regular-immersion setting [[1](#), Prop. 2.6, pp. 43–44; Thm. 6.2, pp. 98–99].

When D is not Cartier, no such local equation need exist. For example, suppose that near a boundary point the ideal sheaf of D is generated by several equations

$$I_D = (t_1, t_2) \subset \mathcal{O}_X,$$

and that I_D fails to be locally principal. Then the inclusion

$$i : D \hookrightarrow X$$

is generally not a regular immersion of codimension one. The boundary may possess embedded components or singularities, and there is no canonical virtual normal line bundle analogous to the Cartier case.

Now let

$$f : \mathcal{Y} \longrightarrow X$$

be a quasi-smooth test morphism and form the derived boundary

$$\mathcal{D} = \mathcal{Y} \times_X^L D.$$

In the proof of [Theorem 4.1](#), the crucial step is the refined base-change identity on Chow homology

$$i_{\mathcal{Y}}^! f^! (\xi) = f_D^! i^! (\xi), \quad \xi \in \mathrm{CH}_*(X)_{\mathbb{Q}}.$$

For the Hodge class one takes

$$\xi = c_g(\mathbb{E}) \cap [X].$$

If the boundary is not Cartier, the refined Gysin map

$$i^! : \mathrm{CH}_*(X)_{\mathbb{Q}} \rightarrow \mathrm{CH}_{*-1}(D)_{\mathbb{Q}}$$

need not exist in the codimension-one form used here, and the derived base-change argument is therefore unavailable.

Thus the argument proving

$$i_{\mathcal{Y}}^! \left(c_g(Lf^* \mathbb{E}) \cap [\mathcal{Y}]^{\mathrm{vir}} \right) = 0$$

breaks down because the required codimension-one refined Gysin formalism is absent.

This example isolates the role of the Cartier condition in the present argument. It is the hypothesis that supplies the codimension-one regular immersion $i : D \hookrightarrow X$, its corresponding refined Gysin pullback, and the quasi-smooth virtual codimension-one derived base change $i_{\mathcal{Y}}$. No claim is made here that every possible extension of the boundary-vanishing theorem requires the boundary to be Cartier; the claim is only that the specific codimension-one refined Gysin formalism used in [Theorem 4.1](#) relies on this hypothesis.

Remark 7.2 (Role of the quasi-smoothness hypothesis). The quasi-smoothness hypothesis on f should not be interpreted as the source of the quasi-smooth or virtual Cartier nature of the induced boundary morphism

$$i_{\mathcal{Y}} : \mathcal{Y} \times_X^{\mathbf{L}} D \longrightarrow \mathcal{Y}.$$

That property follows from the fact that $i : D \hookrightarrow X$ is a Cartier divisor and is preserved by derived base change.

The quasi-smoothness hypothesis on f instead supplies the virtual pullbacks

$$f^! \quad \text{and} \quad f_D^!$$

in the derived intersection-theoretic framework used in this paper, and permits the base-change identity

$$i_{\mathcal{Y}}^! \circ f^! = f_D^! \circ i^!.$$

Likewise, since $\mathbb{E}$ is locally free on X ,

$$Lf^*\mathbb{E} \simeq f^*\mathbb{E}$$

for arbitrary derived pullback; this fact does not require quasi-smoothness of f . Thus quasi-smoothness is used for the virtual-pullback part of the argument, not for the pullback of the Hodge bundle and not for the derived Cartier property of the boundary.

8 Arithmetic Boundary Vanishing

We work in the framework of arithmetic Chow groups and arithmetic intersection products of Gillet–Soulé, in which arithmetic cycles are represented by algebraic cycles together with Green currents and the resulting arithmetic Chow groups carry product and functoriality structures [\[10, Secs. 3.3, 4.2–4.4\]](#). The result below is conditional on the existence of arithmetic refined pullbacks for the derived boundary square and on their compatibility with arithmetic Chern classes and arithmetic refined base change.

Let $X = \mathcal{A}_g^{\text{tor}}$ be an arithmetic toroidal compactification over $\text{Spec } \mathbb{Z}$, with boundary $i : D \hookrightarrow X$. Let $\overline{\mathbb{E}}$ be the extended Hodge bundle equipped with an admissible hermitian metric, and write

$$\widehat{c}_g(\overline{\mathbb{E}}) \in \widehat{\text{CH}}^g(X)_{\mathbb{Q}}.$$

Theorem 8.1 (Conditional arithmetic preservation of boundary vanishing). *Let $f : \mathcal{Y} \rightarrow X$ be an quasi-smooth derived base change and put*

$$\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D.$$

Assume that:

1. *arithmetic refined pullbacks*

$$\widehat{i}^!, \quad \widehat{i}_{\mathcal{Y}}^!, \quad \widehat{f}^!, \quad \widehat{f}_D^!$$

are defined in the chosen arithmetic Chow theory;

2. *arithmetic Chern classes are compatible with the arithmetic virtual pullback f , so that*

$$\widehat{f}^! \left(\widehat{c}_g(\overline{\mathbb{E}}) \cap [X] \right) = \widehat{c}_g(Lf^*\overline{\mathbb{E}}) \cap [\mathcal{Y}]^{\text{vir}};$$

3. *the arithmetic refined base-change identity*

$$\widehat{i}_{\mathcal{Y}}^! \widehat{f}^! = \widehat{f}_D^! \widehat{i}^!.$$

holds for the derived boundary square.

If

$$\widehat{i}^! \left(\widehat{c}_g(\overline{\mathbb{E}}) \cap [X] \right) = 0,$$

then

$$\widehat{i}_{\mathcal{Y}}^! \left(\widehat{c}_g(Lf^*\overline{\mathbb{E}}) \cap [\mathcal{Y}]^{\text{vir}} \right) = 0.$$

Proof. By the assumed compatibility of arithmetic Chern classes with arithmetic virtual pullback,

$$\hat{f}^! \left(\hat{c}_g(\overline{\mathbb{E}}) \cap [X] \right) = \hat{c}_g(Lf^*\overline{\mathbb{E}}) \cap [\mathcal{Y}]^{\text{vir}}.$$

Applying the arithmetic refined boundary pullback gives

$$\hat{i}_{\mathcal{Y}}^! \left(\hat{c}_g(Lf^*\overline{\mathbb{E}}) \cap [\mathcal{Y}]^{\text{vir}} \right) = \hat{i}_{\mathcal{Y}}^! \hat{f}^! \left(\hat{c}_g(\overline{\mathbb{E}}) \cap [X] \right).$$

Using the assumed arithmetic refined base-change identity,

$$\hat{i}_{\mathcal{Y}}^! \hat{f}^! = \hat{f}_D^! \hat{i}^!,$$

this becomes

$$\hat{f}_D^! \left(\hat{i}^! \left(\hat{c}_g(\overline{\mathbb{E}}) \cap [X] \right) \right).$$

The latter class is zero by the arithmetic refined boundary-vanishing hypothesis. Hence

$$\hat{i}_{\mathcal{Y}}^! \left(\hat{c}_g(Lf^*\overline{\mathbb{E}}) \cap [\mathcal{Y}]^{\text{vir}} \right) = 0.$$

□

Corollary 8.2 (Functorial arithmetic refined boundary pullback). *Under the hypotheses of Theorem 8.1, the refined arithmetic boundary cycle of the pulled-back top Hodge class is functorially determined by the original refined arithmetic boundary cycle through*

$$\hat{i}_{\mathcal{Y}}^! \left(\hat{c}_g(Lf^*\overline{\mathbb{E}}) \cap [\mathcal{Y}]^{\text{vir}} \right) = \hat{f}_D^! \left(\hat{i}^! \left(\hat{c}_g(\overline{\mathbb{E}}) \cap [X] \right) \right).$$

In particular, if

$$\hat{i}^! \left(\hat{c}_g(\overline{\mathbb{E}}) \cap [X] \right) = 0,$$

then

$$\hat{i}_{\mathcal{Y}}^! \left(\hat{c}_g(Lf^*\overline{\mathbb{E}}) \cap [\mathcal{Y}]^{\text{vir}} \right) = 0$$

for every admissible quasi-smooth derived test family.

Proof. The asserted identity is the arithmetic refined base-change identity of Theorem 8.1 applied to

$$\hat{c}_g(\overline{\mathbb{E}}) \cap [X],$$

together with the assumed compatibility

$$\hat{f}^! \left(\hat{c}_g(\overline{\mathbb{E}}) \cap [X] \right) = \hat{c}_g(Lf^*\overline{\mathbb{E}}) \cap [\mathcal{Y}]^{\text{vir}}.$$

□

Remark 8.3. The theorem is not used in the proof of the geometric results above. Its role is to isolate the exact arithmetic input needed to transport the derived boundary-vanishing argument to arithmetic Chow theory. Thus the non-formal arithmetic content is contained in the existence and base-change compatibility of the arithmetic refined pullbacks, not in the final vanishing implication.

9 Conclusion

We proved that refined boundary vanishing of the top Chern class of the Hodge bundle on a toroidal compactification is preserved under quasi-smooth derived base change. The central mechanism is the derived boundary square

$$\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D,$$

together with refined/virtual Gysin base-change compatibility on Chow homology. This formulation is distinct from ordinary operational Chow pullback, whose contravariant functoriality is not at issue.

The result gives a compact principle for transporting refined tautological boundary-vanishing statements from $\mathcal{A}_g^{\text{tor}}$ to derived test spaces, complete-intersection families, and non-Tor-independent boundary intersections. In the latter case the derived boundary packages the higher Tor data intrinsically, while the corresponding classical refined intersection theory is governed by the usual excess-intersection formalism.

References

- [1] W. Fulton, *Intersection Theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 2, Springer-Verlag, Berlin, 1998.
- [2] G. Faltings and C.-L. Chai, *Degeneration of Abelian Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 22, Springer-Verlag, Berlin, 1990.
- [3] D. Mumford, *Hirzebruch's proportionality theorem in the non-compact case*, Invent. Math. **42** (1977), 239–272.
- [4] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), no. 3, 495–536. doi:10.1007/s002220050351.
- [5] B. Toën and G. Vezzosi, *Homotopical Algebraic Geometry II: Geometric Stacks and Applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902.
- [6] C. Manolache, *Virtual pull-backs*, J. Algebraic Geom. **21** (2012), no. 2, 201–245.
- [7] A. A. Khan, *Virtual fundamental classes of derived stacks I*, arXiv:1909.01332v1 [math.AG], 2019.
- [8] J. Lurie, *Higher Algebra*, version of September 18, 2017, available at the author's webpage.
- [9] D. Gaitsgory and N. Rozenblyum, *A Study in Derived Algebraic Geometry. Vol. II: Deformations, Lie Theory and Formal Geometry*, Mathematical Surveys and Monographs, Vol. 221, American Mathematical Society, Providence, RI, 2017.
- [10] H. Gillet and C. Soulé, *Arithmetic intersection theory*, Publ. Math. Inst. Hautes Études Sci. **72** (1990), 93–174.
- [11] J. I. Burgos Gil, J. Kramer, and U. Kühn, *Arithmetic characteristic classes of automorphic vector bundles*, Doc. Math. **10** (2005), 619–716.