

Prime-to- p Isogenies, Component Groups, and Cohomological Inertia for Néron Models

Kundnani Rahul Thakurdas¹, Vinoth Marimuthu², Dr. Khursheed Alam^{3,*}, Dr. Shri Kant Ojha⁴

¹Ph.D. Research Scholar, Department of Mathematics, Sharda University, Greater Noida, India

²University of Colorado Boulder, Boulder, CO 80309, United States

³Professor and Head, Department of Mathematics & Data Science, Sharda University, Greater Noida, India

⁴Professor, Department of Mathematics & Data Science, Sharda University, Greater Noida, India

*Corresponding Author: khursheed.alam@sharda.ac.in

Abstract We study Néron models in arithmetic families over regular integral bases of finite type over $S = \operatorname{Spec} \mathbb{Z}$, with $\operatorname{Spec} \mathbb{Z}$ as the basic motivating case. The purpose is to isolate a common tame prime-to-residue-characteristic locus on which finite isogeny, component-group, inertia, Hecke, and height-theoretic data can be compared without confusing standard Néron-model input with the paper’s organizational contribution. The method is to work after explicit shrinking to dense opens, interpret higher-dimensional bases through codimension-one DVRs, and combine Néron-model functoriality, finite-étale prime-to- p kernels, Raynaud semistable reduction, and rational ℓ -adic Tate-module comparison. The principal results show that, for a fixed finite class of degree- m prime-to- p isogenies, the induced maps on identity components have finite étale kernels killed by m , while the maps on component groups have kernel and cokernel annihilated by a bounded power of m . Consequently, the corresponding Tamagawa-index and local geometric defect terms are uniformly controlled on the chosen tame locus. The inertia-invariant subspaces of $H_{\text{ét}}^1$ are preserved under these isogenies, and the Artin conductor exponent itself is unchanged because rational ℓ -adic Tate modules of isogenous abelian varieties are isomorphic. For modular curves, the same mechanism applies to explicitly defined Hecke-isogeny endomorphisms of fixed T_ℓ -stable abelian quotients. For Hodge-type Shimura varieties, the corresponding statement remains conditional on the stated integral-model extension and prime-to-residue-characteristic isogeny hypotheses. The height discussion records the standard functoriality of the Néron–Tate height under isogeny together with precisely defined uniform control of the associated tame geometric data; it does not assert an independent numerical bound for individual local-height contributions or a new global height gap. The practical implication is a precise, reusable tame framework for separating component-group variation from conductor invariance in arithmetic families.

Keywords: Néron model, component group, Tamagawa number, prime-to- p isogeny, semicontinuity, étale cohomology, inertia invariants, Artin conductor, Hecke correspondence (modular curves; Hodge-type under hypotheses), modular curve, Shimura variety

1 Introduction and Main Results

Motivation

Néron models sit at the interface of algebraic geometry and arithmetic: they extend abelian varieties across nonarchimedean places while preserving the universal mapping property that governs integral points and specialization. Over the arithmetic base $S = \operatorname{Spec}(\mathbb{Z})$ (see [Notation 2.1](#)), they encode reduction data through the identity component A^0 , the component groups Φ_p at primes p , and the invariant differential line $\omega_{A/S}$ ([Definition 2.6](#) and [construction 2.11](#)). These geometric pieces control the shape of local Euler factors and Tamagawa numbers, while conductor exponents are defined by the associated ℓ -adic inertia representations ([Definition 2.13](#) and [proposition 2.14](#)); they also interact with global height theory, the Mordell–Weil group $A(\mathbb{Q})$, and rational points on curves mapping to A . While the foundational existence and formal properties are classical—see [\[1\]](#) for Néron models and component groups,

[\[2, Introduction, pp. V–VI; Exposé XIII, pp. 82–115; Exposé XV, pp. 165–196; Exposé XVIII, pp. 254–327\]](#) for vanishing cycles, Picard–Lefschetz theory, and Lefschetz pencils, [\[3, Foreword, pp. v–vii; Ch. II, §§1–6, pp. 33–52\]](#) for degenerations of abelian varieties, [\[4, §§7–8, pp. 112–117; §§15–16, pp. 128–137; §20, pp. 145–150\]](#) for projectivity, isogenies, étale cohomology/pairings, and abelian schemes, [\[5, Theorem 1, p. 493; Theorem 2, pp. 496–497\]](#) for the Néron–Ogg–Shafarevich criterion and potential good reduction, [\[6\]](#) for the basic local harmonic-analysis background used in Euler factors ([\[6, §§2.2–2.4, pp. 308–316\]](#)), and [\[7\]](#) for a detailed modern discussion of Néron-model techniques—arithmetic applications often require carefully organized tame

prime-to-residue-characteristic control across selected arithmetic families—especially along finite isogeny and Hecke correspondences on integral models of modular/Shimura varieties. This paper records such control on explicit dense opens and separates the resulting component-group and inertia-invariant consequences from the standard background input on Néron models, Tate modules, and height functoriality.

Global framing and scope. Although $\text{Spec } \mathbb{Z}$ serves as the basic arithmetic base and motivating example, all results asserting variation, semicontinuity, or uniform bounds are formulated for abelian schemes or Néron models varying in arithmetic families A/T with T a regular integral scheme of finite type over $\text{Spec } \mathbb{Z}$. No theorem claims the existence of infinitely many additive fibers for a fixed abelian variety over $\mathbb{Q}$.

Novelty and separation from standard input

The standard inputs used below are the existence and functoriality of Néron models over Dedekind bases, the Raynaud description of semistable reduction, the finite-étale behavior of prime-to-residue-characteristic kernels, and the isomorphism of rational ℓ -adic Tate modules under isogeny ([8, §6, Proposition 6.1 and Tate-module discussion, pp. 30–32]). The new contribution of this paper is not a new existence theorem for Néron models, but the systematic packaging of these standard tools on one fixed tame codimension-one locus and their simultaneous use for component groups, inertia invariants, Hecke correspondences, and local height terms.

More precisely, the paper contributes:

- (i) a common-open formulation for fixed finite families of degree- m prime-to- p isogenies;
- (ii) uniform kernel-cokernel annihilation control on component groups;
- (iii) an explicit separation between Tamagawa/component-group defect control and Artin-conductor invariance;
- (iv) an explicit Hecke-endomorphism statement for fixed T_ℓ -stable modular quotients, together with a separate conditional Hodge-type statement under integral-model extension hypotheses;
- (v) the standard functoriality of the Néron–Tate height under isogeny, together with the precisely defined tame geometric control of Definition 4.6, without claiming a new global positive height lower bound.

Outline of results

The logical flow is as follows; each entry records the downstream arithmetic consequences and a representative example.

- **Existence/uniqueness in arithmetic families.** *Theorem 3.1 packages the standard codimension-one Néron-model extension and functoriality facts needed later for prime-to- p isogenies. Its role is to fix the precise tame/Dedekind scope in which component-group maps*

are controlled; it is not used as a new global Néron-model theorem over higher-dimensional bases.

- **Component groups and constructibility.** *Theorem 3.5 constructs a common tame open $U \subseteq T^\circ$ on which the relevant component-group sheaf is finite étale. Consequently,*

$$t \mapsto v_q\left(\#\Phi_{A,t}\left(\overline{\kappa(t)}\right)\right) = v_q(\bar{c}_{A,t})$$

is locally constant and hence formally upper semicontinuous. By contrast,

$$c_{A,t} = \#\Phi_{A,t}(\kappa(t))$$

is the order of the Frobenius-fixed subgroup and is not asserted to be locally constant without an additional hypothesis controlling the Frobenius action. The real content is the common-open and kernel-cokernel control for the fixed finite class of prime-to-residue-characteristic isogenies. Lemma 3.7 records constructibility of geometric component-group orders beyond this tame open.

- **Cohomological invariants under base change and isogeny.** *In Theorem 4.1 we compare $H_{\text{ét}}^1(\cdot)^{I_p}$ across the family and along isogenies (Items (i) to (iii)), implying rational local L -factor invariance and conductor stability in Proposition 6.1 and corollary 6.2. This is illustrated by Examples 4.4 and 4.5.*

- **Hecke endomorphisms (precise scope).** For a modular Jacobian $J_0(N)$, the two degeneracy maps

$$\pi_1, \pi_2: X_0(N\ell) \longrightarrow X_0(N)$$

define the Hecke endomorphism

$$T_\ell = (\pi_1)_* \pi_2^* \in \text{End}_{\mathbb{Q}}(J_0(N)).$$

After passing to a T_ℓ -stable abelian quotient A on which the induced endomorphism u_ℓ is an isogeny, the ordinary prime-to-residue-characteristic Néron-model formalism applies to u_ℓ . The resulting conclusions concern the action of this fixed endomorphism on the Néron model, component groups, and rational ℓ' -adic cohomology of the fixed abelian variety A ; they do not assert variation across a family of Néron models.

- **Heights.** Propositions 4.7 and 6.5 record the standard functoriality

$$\widehat{h}_{\varphi^*L}(P) = \widehat{h}_L(\varphi(P))$$

and the precisely defined tame geometric control of Definition 4.6. No independent numerical bound for individual local-height contributions or new global positive lower bound across arbitrary isogeny classes is asserted.

2 Preliminaries

Throughout, we work over the base scheme $\text{Spec } \mathbb{Z}$. Unless otherwise indicated, all schemes are separated and of finite type over their base. For general references: we use [9] for scheme-theoretic conventions; [10, Exposés I–III] for étale cohomology; [2, Exposé XIII, pp. 82–115; Exposé XIV, pp. 116–164] for the vanishing-cycles formalism and comparison with the transcendental theory; [1] for Néron models, component groups, and smoothening; and [4, §§1–3, pp. 104–107; §§7–8, pp. 112–117; §20, pp. 145–150] and [3, Ch. I, §2.3, p. 8; Ch. II, §1, p. 33] for background on abelian varieties and their degenerations.

All background results stated in this section are standard; any novel contributions begin in later sections.

2.1 Schemes and the arithmetic base $\text{Spec } (\mathbb{Z})$

Notation 2.1 (Arithmetic base). We denote by $S = \text{Spec } \mathbb{Z}$ the base scheme. For a prime p we write $S_p = \text{Spec } \mathbb{Z}_{(p)}$ and $\kappa(p) = \mathbb{F}_p$ for its residue field. The generic point is $\eta = \text{Spec } \mathbb{Q}$. If X/S is a scheme, we denote by X_η the generic fiber and by X_p the special fiber over $\kappa(p)$.

Notation 2.2 (Indices for points and primes). Throughout the paper we *reserve* the following symbols:

- t denotes a closed point when the base is Dedekind; when a higher-dimensional regular base is considered, local Néron-model statements are made only after restriction to a specified codimension-1 point.
- p denotes the positive residue characteristic attached to the relevant local point t , namely $p = \text{char } \kappa(t)$.
- q denotes an *auxiliary rational prime* used only as a valuation index in expressions like $v_q(\# \Phi_{A,t})$.

Remark 2.3 (Standing scope and hypothesis convention). Throughout the paper we repeatedly shrink to dense opens. To make every later statement checkable at the level of hypotheses, we adopt the following convention.

(S2) Fixed tame/étale locus for a chosen degree m . Whenever an integer $m \geq 1$ is fixed, we shrink to a dense open on which: (i) m is invertible, and (ii) the Néron model is semistable (semistable) so that vanishing-cycle formalism applies ([2, Exposé XIII, pp. 82–115]). We then further shrink to a dense open $U = U(m)$ on which every degree- m prime-to-residue-characteristic isogeny considered has kernel extending to a *finite étale* subgroup scheme (equivalently, all such kernels are finite étale on U).

(S3) Quantifiers. Accordingly, phrases such as “for every degree- m isogeny” always mean: *for every degree- m isogeny whose kernel is finite étale over the fixed open $U = U(m)$* . Whenever U is not the same as the one fixed earlier, it will be explicitly renamed.

Remark 2.4 (Global vs. local arithmetic). The structure of X over $\text{Spec } \mathbb{Z}$ reflects both global information (e.g. height pairings, discriminants) and local invariants (e.g. component

groups at p). Connecting these perspectives is central for the arithmetic applications developed later.

Lemma 2.5 (Flat base change for smooth morphisms). *Let $X \rightarrow S$ be smooth and $S' \rightarrow S$ flat. Then the base change $X \times_S S' \rightarrow S'$ is smooth, and formation of the sheaf of differentials $\Omega_{X/S}^1$ commutes with base change.*

Proof. The compatibility of differentials with base change follows from EGA IV, §16.5, especially (16.5.10.1) and Proposition 16.5.11 [9, §16.5, Prop. 16.5.11, pp. 29–30]. The stability of smoothness under base change is standard in EGA IV. $\square$

2.2 Néron models: definitions and properties

Definition 2.6 (Néron model). Let K be the function field of S and A/K an abelian variety. A *Néron model* of A over S is a smooth separated group scheme A/S of finite type such that $A_\eta \cong A$ and satisfying the *Néron mapping property*: for every smooth S -scheme Y and every K -morphism $f_\eta : Y_\eta \rightarrow A_\eta$, there exists a unique S -morphism $f : Y \rightarrow A$ extending f_η ([1, Ch. 1, §1.2, Def. 1, p. 12]).

Lemma 2.7 (Existence of Néron models). *If A/K is an abelian variety, then a Néron model A/S exists and is unique up to unique isomorphism.*

Proof. This is classical, see [1, Ch. 1, §1.3, Thm. 1 and Cor. 2, p. 16; Ch. 1, §1.4, Prop. 1, p. 18]. $\square$

Proposition 2.8 (Component groups). *Let A/S be a Néron model of an abelian variety A/K . For each closed point $p \in S$, denote by $\Phi_p = A_p/A_p^0$ the component group of the special fiber. Then Φ_p is a finite étale group scheme over $\kappa(p)$, and the Tamagawa number at p is given by $c_p = \# \Phi_p(\kappa(p))$.*

Proof. See [1]. $\square$

Notation 2.9 (Geometric component order versus Tamagawa number). For every closed point t , fix a geometric point $\bar{t} = \text{Spec } \overline{\kappa(t)}$ above t , and write

$$\bar{c}_{A,t} := \# \Phi_{A,t}(\overline{\kappa(t)})$$

for the geometric order of the component group. The local Tamagawa number is

$$c_{A,t} := \# \Phi_{A,t}(\kappa(t)).$$

Thus $\bar{c}_{A,t}$ and $c_{A,t}$ are distinct invariants in general: $\bar{c}_{A,t}$ is the cardinality of the geometric fiber, whereas $c_{A,t}$ is the number of Frobenius-fixed geometric components.

Remark 2.10 (Relation to reduction type). Good reduction corresponds to Φ_p trivial, while semistable reduction means ([11, Exposé IX, §3]) that the identity component $(A_p)^0$ of the special fiber is *semiabelian* (in particular, it contains a toric part). The component group $\Phi_p := A_p/(A_p)^0$ is always a finite étale group scheme over $\kappa(p)$, and remains finite étale in both good and semistable reduction. Thus:

$$\text{Good reduction} \Rightarrow \Phi_p \text{ trivial,}$$

and

Semistable reduction $\Rightarrow (A_p)^0$ semiabelian (with torus part),

while Φ_p is finite étale.

This corrects the common shorthand “ Φ_p a torus,” since the toric part lies in the identity component rather than in the component group.

Construction 2.11 (Sheaf of differentials). Let A/S be a Néron model. Define $\omega_{A/S} = e^*(\Omega_{A/S}^1)$, the pullback of the sheaf of differentials along the identity section $e : S \rightarrow A$. This invertible sheaf controls height pairings and appears naturally in Arakelov intersection theory and in the study of heights of abelian varieties [12, §§2–3, pp. 350–355].

2.3 Cohomological background

Lemma 2.12 (Localization for abelian schemes; semistable Raynaud description). *Let p be a rational prime, $K = \mathbb{Q}_p$, $\mathbb{Z}_p$ its ring of integers, and $\mathbb{F}_p$ its residue field. (Proper smooth case.) If $A/\mathbb{Z}_p$ is an **abelian scheme** (i.e. proper and smooth), then for every prime $\ell \neq p$ and every $i \geq 0$ there is a canonical specialization isomorphism*

$$H_{\text{ét}}^i(A_{\overline{K}}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^i(A_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell),$$

functorial and compatible with G_K (see also [10, Exposé XVI, Théorème 1.2]). (Semistable case.) If A/K has semistable reduction and $A_{\mathbb{F}_p}^0$ fits into the Raynaud extension

$$0 \rightarrow T \rightarrow A_{\mathbb{F}_p}^0 \rightarrow B \rightarrow 0,$$

(as in [11, Exposé IX, §3.6, pp. 390–399]), with T a torus of rank τ_p and B an abelian variety of dimension a , then

$$\dim_{\mathbb{Q}_\ell} V_\ell(A)^{I_p} = 2a + \tau_p = 2g - \tau_p.$$

This statement concerns the I_p -invariants of the generic-fiber Tate module. It should not be read as an unrestricted comparison theorem for étale cohomology of an arbitrary non-proper Néron model.

Proof. In the proper and smooth (abelian–scheme) case, the claimed specialization isomorphism is the standard one furnished by smooth and proper base change for $A/\mathbb{Z}_p$ together with the Néron–Ogg–Shafarevich criterion for $\ell \neq p$ ([5, Theorem 1, p. 493]).

In the semistable case, the asserted formula is the standard Raynaud-extension description of the inertia invariants of the generic-fiber Tate module. More precisely, when the identity component of the special fiber fits into $0 \rightarrow T \rightarrow A_{\mathbb{F}_p}^0 \rightarrow B \rightarrow 0$, the invariant part has dimension $2 \dim B + \text{rank } T = 2g - \tau_p$. This is only a statement about $V_\ell(A)^{I_p}$ in the semistable/Raynaud-extension setting, not a nearby-cycle comparison theorem for arbitrary non-proper Néron models. $\square$

Definition 2.13 (Conductor exponent). Let A/K be an abelian variety over a nonarchimedean local field with residue characteristic p , and let $\ell \neq p$. Write

$$V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

and let I_p denote the inertia subgroup.

The conductor exponent $f_p(A)$ is the Artin conductor exponent of the ℓ -adic representation $V_\ell(A)$. Explicitly,

$$f_p(A) = \text{codim}_{\mathbb{Q}_\ell} V_\ell(A)^{I_p} + \text{Swan}(V_\ell(A)).$$

Equivalently, using $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell) \simeq V_\ell(A)^\vee$,

$$f_p(A) = \text{codim}_{\mathbb{Q}_\ell} H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)^{I_p} + \text{Swan}(H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)).$$

In particular, the Swan conductor is only the wild part of $f_p(A)$, not the entire conductor exponent.

Proposition 2.14 (Discriminant and conductor in the semistable case). *Let E/K be an elliptic curve over a nonarchimedean local field, let $\Delta_{\min}$ be the discriminant of a minimal Weierstrass equation, and let $f_p(E)$ be the local conductor exponent.*

Assume that E has semistable reduction. Then exactly one of the following occurs:

(i) *If E has good reduction, then*

$$\text{ord}_p(\Delta_{\min}) = 0 \quad \text{and} \quad f_p(E) = 0.$$

(ii) *If E has split or nonsplit multiplicative reduction of Kodaira type I_n , with $n \geq 1$, then*

$$\text{ord}_p(\Delta_{\min}) = n \quad \text{and} \quad f_p(E) = 1.$$

Consequently, for semistable elliptic curves,

$$\text{ord}_p(\Delta_{\min}) \geq f_p(E),$$

and equality holds precisely in the good-reduction case and in the multiplicative case of Kodaira type I_1 . In particular, equality does not hold for arbitrary semistable reduction.

Proof. If E has good reduction, a minimal Weierstrass equation has unit discriminant, so

$$\text{ord}_p(\Delta_{\min}) = 0.$$

The inertia action on $V_\ell(E)$ is trivial, and therefore

$$f_p(E) = 0.$$

Suppose instead that E has split or nonsplit multiplicative reduction. Its Kodaira symbol is I_n for a uniquely determined integer $n \geq 1$, and the Kodaira–Néron classification gives

$$\text{ord}_p(\Delta_{\min}) = n.$$

The inertia representation is tamely ramified and unipotent with a one-dimensional invariant subspace. Hence

$$\text{codim}_{\mathbb{Q}_\ell} V_\ell(E)^{I_p} = 1 \quad \text{and} \quad \text{Swan}(V_\ell(E)) = 0.$$

Thus

$$f_p(E) = 1.$$

It follows that

$$\text{ord}_p(\Delta_{\min}) \geq f_p(E)$$

in the semistable cases. In the good-reduction case both sides are zero, while in the multiplicative case equality is equivalent to $n = 1$. $\square$

Remark 2.15 (Semistability does not imply discriminant-conductor equality). For a semistable elliptic curve with multiplicative reduction of type I_n , one has

$$f_p(E) = 1 \quad \text{but} \quad \text{ord}_p(\Delta_{\min}) = n.$$

Thus, for example, reduction of type I_5 gives

$$\text{ord}_p(\Delta_{\min}) = 5 \neq 1 = f_p(E).$$

Any later use of [Proposition 2.14](#) must therefore distinguish conductor invariance from variation of the minimal-discriminant valuation.

Remark 2.16 (No unboundedness assertion for additive fibers). We do not use any assertion that infinitely many additive fibers force unbounded component groups. In small dimensions, especially for elliptic curves, additive Kodaira component groups are bounded in small finite sets. Any growth statement in this paper concerns explicitly specified families and explicitly tracked invariants, not a general unboundedness theorem for additive reduction.

3 Structural Properties of Néron Models

We fix the base and notation from [Notation 2.1](#). All background on existence, identity components, and component groups is in [Definition 2.6](#), [lemma 2.7](#), [proposition 2.8](#), and [remark 2.10](#). In this section we establish structural results for Néron models in arithmetic families and under isogeny, and we record consequences for component groups and local geometric defect terms. The Artin conductor defined in [Definition 2.13](#) is treated separately through the rational ℓ -adic representation, while [Proposition 2.14](#) records the distinct behavior of the minimal discriminant in the semistable elliptic case.

3.1 Existence and uniqueness in arithmetic families

Theorem 3.1 (Codimension-one Néron-model control and tame isogeny functoriality). *Let T be a regular integral Dedekind scheme of finite type over $\text{Spec } \mathbb{Z}$, with generic point η_T , and let $A_{\eta_T}/K(T)$ be an abelian variety. After replacing T by a nonempty open subscheme T° , the following assertions hold. For a higher-dimensional regular arithmetic base, this theorem is used only after restriction to the corresponding codimension-one DVR, as in [Remark 2.3](#).*

(i) **(Relative Néron mapping property)**. *For every smooth T° -scheme Y and every $K(T)$ -morphism $Y_{\eta_T} \rightarrow A_{\eta_T}$ there is a unique T° -morphism $Y \rightarrow A_{T^\circ}$ extending it; thus A_{T° is the Néron model of A_{η_T} over T° .*

(ii) **(Minimality among smooth separated models)**. *For any smooth separated group scheme $G \rightarrow T^\circ$ whose generic fiber is isomorphic to A_{η_T} , there exists a unique T° -morphism $G \rightarrow A_{T^\circ}$ restricting to the identity on the generic fiber.*

(iii) **(Compatibility with étale base change)**. *For any étale morphism $T' \rightarrow T^\circ$ with T' regular and integral, the pullback $(A_{T^\circ})_{T'}$ is the Néron model of $(A_{\eta_T})_{K(T')}$ over T' . **Hypotheses for (iv)**. Fix an integer $m \geq 1$ and let*

$$\varphi_{\eta_T}: A_{\eta_T} \longrightarrow B_{\eta_T}$$

be an isogeny of degree m . After possibly shrinking T° further, set $U := T^\circ$ and assume that

(H1) *m is invertible on U , and every fiber of $A_U \rightarrow U$ and $B_U \rightarrow U$ is semiabelian (i.e. both abelian varieties have semistable reduction on U ; see [11, Exposé IX, §3]);*

(H2) *the generic kernel $\ker(\varphi_{\eta_T})$ extends to a finite étale subgroup scheme of A_U over U .*

Under (H1)–(H2), the vanishing-cycle formalism ([2, Exposé XIII, pp. 82–115]) applies and the induced morphism on component groups is well defined.

(iv) **(Isogeny functoriality and component-group control on the tame prime-to- p locus)**

Under (H1)–(H2), the isogeny φ_{η_T} extends uniquely to a morphism of the corresponding Néron models over U . For every closed point $t \in U$:

- the induced morphism on identity components*

$$(A_t)^0 \longrightarrow (B_t)^0$$

is an isogeny of semiabelian varieties whose kernel is finite étale and killed by m . In the abelian-scheme case its degree divides m .

- the induced morphism on component groups*

$$\Phi_{A,t} \longrightarrow \Phi_{B,t}$$

has kernel and cokernel killed by m , hence their orders divide a power of m . No assertion is made that their orders divide m , and no divisibility formula for $c_{B,t}/c_{A,t}$ is claimed without a separate finite-group calculation.

Remark 3.2 (Scope of [Theorem 3.1](#)). The global Néron mapping property is used only over Dedekind bases. For higher-dimensional regular bases, all invocations are understood after restriction to codimension-one DVRs, as fixed in [Remark 2.3](#).

Scope. The ingredients used in this theorem are standard consequences of Néron-model functoriality, semistable reduction, and finite-étale prime-to-residue-characteristic isogeny theory. The contribution here is not a new existence theorem for Néron models over higher-dimensional bases, but the precise packaging of these standard inputs on a fixed tame codimension-one locus in a form suitable for the component-group, cohomological, and Hecke-uniformity applications below.

Proof. Since T° is Dedekind, the existence and uniqueness of the Néron model A_{T° follow from the standard Néron-model existence and smoothening theory [1, Ch. 1, §1.3; Ch. 3, §3.1].

For (iii), let $T' \rightarrow T^\circ$ be étale. Since the composite of a smooth T' -scheme with the étale morphism $T' \rightarrow T^\circ$ is smooth over T° , the Néron mapping property gives the Néron mapping property after base change to T' . Thus Néron models commute with the stated étale base change. In higher dimension the same argument is used only after restriction to the codimension-1 DVRs.

For (iv), let φ_{η_T} be an isogeny of degree m with $(m, \text{char } \kappa(t)) = 1$ for all $t \in T^\circ$. Functoriality of the Néron mapping property, applied to the prime-to-residue-characteristic isogeny, yields a unique extension

$$\varphi: A_{T^\circ} \longrightarrow B_{T^\circ}$$

in the Dedekind/codimension-one sense [1, Ch. 7, §7.3, Prop. 6, p. 180]. On the fixed tame locus the kernel on the identity component is finite étale and killed by m , since prime-to-residue-characteristic torsion is finite étale in this setting [1, Ch. 7, §7.3, Lem. 2, p. 179]. Passing to the special fiber and using the standard component-group sequence for morphisms of semiabelian/Néron models over DVRs, one obtains an induced map

$$\Phi_{A,t} \longrightarrow \Phi_{B,t}$$

whose kernel and cokernel are killed by m . Therefore their orders divide some power of m , with exponent bounded by the ranks of the finite component groups occurring on the fixed tame family. We do not infer $\# \ker \mid m$, $\# \text{coker} \mid m$, or a divisibility relation between $c_{A,t}$ and $c_{B,t}$.

More explicitly, over the strict henselization of the DVR at t , the exact sequence

$$0 \longrightarrow A_t^0 \longrightarrow A_t \longrightarrow \Phi_{A,t} \longrightarrow 0$$

and its analogue for B , together with the snake lemma applied to the finite-étale prime-to- p isogeny on identity components, gives that both the kernel and cokernel on geometric component groups are annihilated by m . This is the standard BLR/Raynaud component-group functoriality for morphisms of semiabelian Néron models over DVRs.

Note. Throughout, when $\dim T > 1$ the assertions above are understood *fiberwise in codimension 1* (Néron lft-model sense; see [1]); no global mapping property over higher-dimensional regular bases is used. The extension of prime-to- p isogenies to Néron models and the finite-étale behavior on identity components are standard in the literature; see [1] for the Néron-model functoriality/extension formalism. The control

of component-group kernels and cokernels by the isogeny degree follows from [1] and the formalism of vanishing cycles therein. These references justify the structural claims of [Theorem 3.1](#) (iv) and delimit the hypotheses (semi-abelian reduction on an open dense locus, kernel scheme finite étale of order m).

Shrinking once to the locus where the m -torsion is finite étale and the fibers are semiabelian yields the fixed open U of [Theorem 3.5](#).

(For the prime-to- p semistable input used here, see [1, Ch. 7, §7.3, Lem. 2, p. 179] for the finite-étale kernel statement and [1, Ch. 7, §7.3, Prop. 6, p. 180] for extension of the isogeny to Néron models.) □

For each closed point $t \in U$, the morphism

$$\Phi_{A,t} \longrightarrow \Phi_{B,t}$$

has kernel and cokernel annihilated by m . Consequently, the associated component-group defect terms are uniformly controlled in the kernel-cokernel sense by the isogeny degree m . No divisibility formula for $c_{B,t}/c_{A,t}$ is asserted here.

The Artin-conductor issue is treated separately in [Theorem 4.1](#); here the conclusion concerns component groups and Tamagawa-index data only.

This component-group control provides the geometric input used later in [Theorem 4.1](#) and in the tame Hecke-uniformity statements of Section 5.

Example 3.3 (Wild additive prime and failure of the tame prime-to- p mechanism). Let $E/\mathbb{Q}$ admit a 2-isogeny

$$\varphi_\eta: E \longrightarrow E'$$

and suppose that E has additive reduction at $p = 2$ (for example Kodaira type II or III). Set

$$T^\circ = S \setminus \{2\}.$$

Then the hypotheses of [Theorem 3.1](#)(iv) hold on T° , but fail at the wild prime $p = 2$.

Scope at the excluded prime. Over $\mathbb{Z}_2$, the finite flat kernel of φ_η has order 2, but it need not be étale on the special fiber. Consequently, the finite-étale identity-component mechanism used in [Theorem 3.1](#)(iv) does not apply at $p = 2$.

Nevertheless, if

$$\widehat{\varphi}_\eta: E' \longrightarrow E$$

denotes the dual isogeny, then

$$\widehat{\varphi}_\eta \circ \varphi_\eta = [2], \quad \varphi_\eta \circ \widehat{\varphi}_\eta = [2].$$

By functoriality of Néron models and their component groups, the corresponding composites on component groups are multiplication by 2. Hence the kernel and cokernel of

$$\Phi_{E,2} \longrightarrow \Phi_{E',2}$$

are annihilated by 2. Thus what fails at $p = 2$ is the finite-étale description of the kernel and the associated identity-component argument, not the abstract component-group annihilation bound.

Cohomological consequence. For $\ell \neq 2$, wild inertia may act nontrivially on

$$H_{\text{ét}}^1(E_{\mathbb{Q}_2}, \mathbb{Q}_\ell),$$

so a nonzero Swan contribution can appear in the conductor formula. Thus the tame comparison statements of [Theorem 4.1](#) and the uniform component-group bounds of [Theorem 3.5](#) are not expected to persist at the wild additive prime.

3.2 Uniform semi-continuity of component indices under isogeny

Lemma 3.4 (Finite-group control for tame component maps). *Let R be a strict henselian DVR with residue characteristic p , and let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of Néron models induced by an isogeny of generic fibers of degree m , with $p \nmid m$. Assume that the induced morphism on identity components has finite étale kernel killed by m . Then the induced morphism on geometric component groups*

$$\Phi_{\mathcal{A}, \bar{k}} \longrightarrow \Phi_{\mathcal{B}, \bar{k}}$$

has kernel and cokernel annihilated by m . Consequently their orders divide a power of m .

This statement concerns geometric component groups over $\bar{k}$. For a finite residue field k ,

$$\Phi_{\mathcal{A}}(k) = \Phi_{\mathcal{A}}(\bar{k})^{\text{Frob}_k}, \quad \Phi_{\mathcal{B}}(k) = \Phi_{\mathcal{B}}(\bar{k})^{\text{Frob}_k}.$$

Thus constancy of geometric component-group order does not imply constancy of the numbers of k -rational components. Any statement about Tamagawa-number constancy requires a separate hypothesis controlling the Frobenius action.

Theorem 3.5 (Constructibility and tame local constancy of geometric component-group orders). *In the Dedekind setting of [Theorem 3.1](#), fix an integer $m \geq 1$ and shrink T° so that:*

- (H1) *m is invertible on T° , equivalently, every residue characteristic on T° is prime to m ;*
- (H2) *$A_{T^\circ} \rightarrow T^\circ$ is semiabelian, so that the component-group and vanishing-cycle bounds apply on T° .*

Let $\mathcal{I}_m$ be a fixed finite collection of degree- m isogenies

$$\varphi_{\eta_T} : A_{\eta_T} \longrightarrow B_{\eta_T}$$

whose kernels extend, after possibly shrinking T° , to finite étale subgroup schemes. Then there exists a Zariski-open dense subscheme

$$U = U(\mathcal{I}_m) \subseteq T^\circ$$

such that the following conclusions hold simultaneously for every $\varphi_{\eta_T} \in \mathcal{I}_m$ and every closed point $t \in U$:

1. *The induced morphism on geometric component groups*

$$\Phi_{A,t}(\overline{\kappa(t)}) \longrightarrow \Phi_{B,t}(\overline{\kappa(t)})$$

has kernel and cokernel annihilated by m . In particular, their orders divide powers of m ; no assertion is made that either order divides m .

2. *Fix a rational prime $q \nmid m$, used only as a valuation index and distinct from $p = \text{char } \kappa(t)$. On U , the component-group scheme $\pi_0(A_U/U)$ is finite étale. Consequently,*

$$t \longmapsto v_q\left(\#\Phi_{A,t}(\overline{\kappa(t)})\right) = v_q(\bar{c}_{A,t})$$

is locally constant and therefore upper semicontinuous on U .

3. *The local Tamagawa number*

$$c_{A,t} = \#\Phi_{A,t}(\kappa(t)) = \#\Phi_{A,t}(\overline{\kappa(t)})^{\text{Frob}_t}$$

is not asserted to be locally constant on U . Such local constancy follows only on a locus on which the relevant Frobenius action, equivalently the finite étale component-group local system up to isomorphism with its arithmetic monodromy action, is constant.

Notation 3.6 (Fixed tame locus). Fix once and for all the dense open $U \subseteq T^\circ$ provided by [Theorem 3.5](#) for the chosen degree m prime to all residue characteristics on T° . All statements in Sections 4–6 that refer to “on a dense open” are henceforth to be read over this fixed U , unless explicitly stated otherwise.

N.B. On the finite-étale locus, the geometric component order

$$\bar{c}_{A,t} = \#\Phi_{A,t}(\overline{\kappa(t)})$$

is locally constant; the corresponding upper-semicontinuity statement is therefore formal. This does not imply local constancy of

$$c_{A,t} = \#\Phi_{A,t}(\kappa(t)),$$

because the Frobenius action on the geometric component group may vary.

Scope. The content of the theorem is the construction of a common tame open for the fixed finite family of prime-to-residue-characteristic isogenies, the resulting kernel–cokernel control on geometric component groups, and local constancy of their geometric orders. No local constancy of Tamagawa numbers is claimed without a separate hypothesis controlling the Frobenius action.

Proof. By [Lemma 2.5](#) we may shrink T° so that every fiber of $A_{T^\circ} \rightarrow T^\circ$ is semiabelian and the identity components form a semiabelian scheme. Let m be prime to all residue characteristics on T° . For any degree- m isogeny φ_{η_T} , its kernel on the generic fiber is a finite flat group scheme of order m . Because $(m, \text{char } \kappa(t)) = 1$ for all $t \in T^\circ$, there exists a dense open $U \subseteq T^\circ$ on which this kernel extends to a finite étale subgroup of $A[m]$, itself finite étale over U [[1](#), Ch. 7, §7.3, Lem. 2, p. 179]. For each closed point $t \in U$, the local ring $\mathcal{O}_{U,t}$ is a DVR. After passage to its strict henselization, the isogeny restricts to an isogeny of semiabelian identity components with finite étale kernel killed by m . Applying [Lemma 3.4](#) gives the asserted kernel–cokernel control on geometric component groups. This gives (i).

Before invoking semicontinuity, we note that $\pi_0(A_U/U)$ is a finite étale group scheme after a further shrinking of U . Indeed, for the smooth separated group scheme A_U/U , the identity component $(A_U)^0$ is open, and the quotient sheaf

$$\pi_0(A_U/U) := A_U/(A_U)^0$$

is represented, after shrinking to the relevant finite-type locus, by a finite étale U -group scheme; see [1]. Therefore the cardinality of its geometric fibers,

$$t \mapsto \#\pi_0(A_U/U)_{\bar{t}} = \#\Phi_{A,t}(\overline{\kappa(t)}),$$

is locally constant. This conclusion concerns geometric fiber cardinality and does not identify it with the number of $\kappa(t)$ -rational points.

For the full base T° , the constructibility of the geometric component-order function

$$t \mapsto v_q\left(\#\Phi_{A,t}(\overline{\kappa(t)})\right) = v_q(\bar{c}_{A,t})$$

is recorded in Lemma 3.7. No closedness assertion for its level sets is made without additional specialization hypotheses.

Lemma 3.7 (Constructibility of geometric component-group orders). *Let T° and $A_{T^\circ} \rightarrow T^\circ$ be as above, and fix a rational prime $q \nmid m$. After stratifying T° into finitely many locally closed subschemes, the function*

$$t \mapsto \bar{c}_{A,t} = \#\Phi_{A,t}(\overline{\kappa(t)})$$

is locally constant on each stratum. Consequently,

$$t \mapsto v_q(\bar{c}_{A,t})$$

is constructible. On the dense open $U \subseteq T^\circ$ where $\pi_0(A_U/U)$ is finite étale, this function is locally constant.

The lemma makes no corresponding local-constancy assertion for

$$c_{A,t} = \#\Phi_{A,t}(\kappa(t)).$$

Proof. The sheaf of connected components

$$\pi_0(A_{T^\circ}/T^\circ)$$

is a constructible étale group sheaf whose geometric fiber at t is

$$\Phi_{A,t,\bar{t}} = \Phi_{A,t} \times_{\kappa(t)} \overline{\kappa(t)};$$

see [1]. Hence

$$t \mapsto \#\Phi_{A,t}(\overline{\kappa(t)})$$

is constructible. Equivalently, after a finite stratification of T° into locally closed subschemes, this geometric fiber cardinality is locally constant on each stratum. Applying v_q gives the constructibility of $t \mapsto v_q(\bar{c}_{A,t})$.

On the dense open U where $\pi_0(A_U/U)$ is finite étale, its geometric fibers have locally constant cardinality. Therefore $t \mapsto v_q(\bar{c}_{A,t})$ is locally constant on U .

By contrast,

$$c_{A,t} = \#\Phi_{A,t}(\kappa(t)) = \#\Phi_{A,t}(\overline{\kappa(t)})^{\text{Frob}_t}$$

depends on the Frobenius-fixed subgroup. Its local constancy does not follow from constancy of geometric fiber cardinality. $\square$

Remark 3.8 (Why geometric order does not determine the Tamagawa number). A finite étale group scheme may have constant geometric order while its number of residue-field-rational points varies with Frobenius. For example, over $\text{Spec } \mathbb{Z}[1/3]$, the finite étale group scheme μ_3 has

$$\#\mu_3(\overline{\mathbb{F}}_p) = 3$$

for every $p \neq 3$, whereas

$$\#\mu_3(\mathbb{F}_p) = \gcd(3, p-1),$$

which equals 3 for $p \equiv 1 \pmod{3}$ and 1 for $p \equiv 2 \pmod{3}$. Accordingly, the finite-étale argument used above proves local constancy only of geometric component-group order, not of the associated Tamagawa number. $\square$

On U , the geometric component orders

$$\bar{c}_{A,t} = \#\Phi_{A,t}(\overline{\kappa(t)})$$

are locally constant, and the maps between geometric component groups induced by the fixed finite collection of prime-to-residue-characteristic isogenies have kernel and cokernel annihilated by m .

The local Tamagawa numbers are instead

$$c_{A,t} = \#\Phi_{A,t}(\kappa(t)) = \#\Phi_{A,t}(\overline{\kappa(t)})^{\text{Frob}_t}.$$

They are controlled pointwise through the Frobenius-equivariant component-group maps, but they are not asserted to be locally constant as t varies. Local constancy of $c_{A,t}$ requires an additional hypothesis making the Frobenius action constant, for example that the component-group scheme becomes a constant finite group scheme on the stratum under consideration.

These statements concern component groups and their arithmetic fixed subgroups. They do not imply a conductor-variation inequality; the Artin conductor exponents are unchanged under the isogenies considered here.

This theorem supplies the geometric input for the cohomological invariance (Theorem 4.1) by ensuring that all wild variation of inertia invariants is confined to a finite set of excluded primes. It also furnishes the analytic base for conductor stability (Corollary 6.2), for the fixed-quotient Hecke-endomorphism statement Theorem 5.3, and, separately, for the conditional Hodge-type statement Theorem 5.10, thereby linking the geometric control of component groups to the corresponding arithmetic consequences.

Example 3.9 (Boundary failure of the finite-étale kernel hypothesis). Retain the curves $E: y^2 = x^3 - x$ and $E': y^2 = x^3 + 4x$. At $p = 2$, the degree of the isogeny is not prime to the residue characteristic, and the kernel need not extend as a finite étale subgroup scheme. Hence the prime-to- p conclusions of Theorem 3.5 do not apply at this prime.

This example should not be read as asserting that the Artin conductor of the rational ℓ -adic Tate module changes under an isogeny. Rather, it illustrates only the failure of the geometric hypotheses used to control identity components and component-group maps. The local Néron-model geometry may fail to be finite étale, but the rational Tate modules of isogenous abelian varieties remain isomorphic.

3.3 Cohomological consequences

By Lemma 2.12 and definition 2.13, the inertia action on $H_{\text{ét}}^1(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ detects the reduction type and the conductor exponent f_p . Theorems 3.1 and 3.5 control the geometric component groups

$$\Phi_{A,p}(\overline{\kappa(p)})$$

and the kernels and cokernels of their maps under the permitted prime-to-residue-characteristic isogenies. The associated local Tamagawa number

$$c_{A,p} = \# \Phi_{A,p}(\kappa(p))$$

is obtained by taking Frobenius-fixed points and is therefore kept separate from the locally constant geometric component order. Combined with Construction 2.11 and the positivity statements used later, this supplies the geometric input for the cohomological comparison results in the next section.

Section 4 will use the structural controls above to compare cohomology under base change and derive arithmetic consequences for local factors and heights.

4 Cohomological Invariants and Applications

Standing convention for this section. We work over the fixed open $U \subseteq T^\circ$ from Theorem 3.5 (Notation 3.6); in particular, all degree- m isogenies considered below have kernel finite étale over U , and the component-group control is uniform on U . All uses of “extension to Néron models” and all quantifiers over degree- m isogenies are understood in the sense of Remark 2.3.

We keep the base $S = \text{Spec } \mathbb{Z}$ and the notation from Notation 2.1. Background on Néron models and component groups was set in Definition 2.6, lemma 2.7, proposition 2.8, remark 2.10, and construction 2.11. Cohomological inertia and the Artin conductor were fixed in Lemma 2.12 and definition 2.13, while Proposition 2.14 separately records the relation between the minimal-discriminant valuation and the conductor exponent for semistable elliptic curves. Structural properties in families and under prime-to-residue-characteristic isogenies were established in Theorems 3.1 and 3.5 and illustrated already by Examples 3.3 and 3.9. Here we quantify how ℓ -adic cohomology and inertia invariants vary in arithmetic families and under isogeny, and we record consequences for local factors and heights.

4.1 Uniform comparison in families and under isogeny

Theorem 4.1 (Refined comparison of cohomology under base change and prime-to- p isogeny). *Let T be a regular integral Dedekind scheme of finite type over S , with generic point η_T and function field $K(T)$. Let $A_{\eta_T}/K(T)$ be an abelian variety of dimension g , and let $A_{T^\circ} \rightarrow T^\circ$ be its Néron model on a nonempty open $T^\circ \subseteq T$ as in Theorem 3.1. Fix a prime ℓ . After possibly shrinking to a dense open $U \subseteq T^\circ$, the following hold for every closed point $t \in U$ with residue characteristic $p = \text{char } \kappa(t)$. For a higher-dimensional regular arithmetic base, the statement is used only after restriction to the corresponding codimension-one DVR as in Remark 2.3:*

- (i) **(Base-change invariance on the abelian-scheme locus; nearby-cycles on the semistable locus)** *On the locus where A_U/U is an abelian scheme, proper and smooth base change yields a canonical isomorphism (see [10, Exposé XVI, Théorème 1.2])*

$$H_{\text{ét}}^i((A_U)_{\eta_T}, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^i((A_U)_{K(t)}, \mathbb{Q}_\ell)^{I_t}.$$

On the semistable (tame) locus for a general Néron model, nearby cycles ([2, Exposé XIII, pp. 82–115]) identify $H_{\text{ét}}^1(A_{K(t)}, \mathbb{Q}_\ell)^{I_t}$ with the cohomology of the semiabelian part of the special fiber, hence for $\ell \neq p$,

$$\dim_{\mathbb{Q}_\ell} H_{\text{ét}}^1(A_{K(t)}, \mathbb{Q}_\ell)^{I_t} = 2g - \tau_t.$$

- (ii) **(Isogeny invariance of rational inertia invariants)** *Let $\varphi_{\eta_T} : A_{\eta_T} \rightarrow B_{\eta_T}$ be an isogeny of degree m with $(m, p) = 1$ on U , and let $\varphi : A_U \rightarrow B_U$ be its extension from Theorem 3.1. Then for $\ell \neq p$ the induced map on I_t -invariants*

$$H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} \xrightarrow{\sim} H_{\text{ét}}^1(B_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t}$$

is an isomorphism. Equivalently, the function $t \mapsto \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t}$ is constructible on the fixed tame semistable locus and invariant under all prime-to- p isogenies of fixed degree m .

- (iii) **(Artin conductor identity and isogeny invariance)** *For $\ell \neq p$, the Artin conductor exponent is*

$$f_{A,t} = \text{codim}_{\mathbb{Q}_\ell} H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} + \text{Swan} \left(H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell) \right).$$

On the semistable locus one has

$$\text{codim}_{\mathbb{Q}_\ell} H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} = \tau_t.$$

Hence

$$f_{A,t} = \tau_t + \text{Swan} \left(H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell) \right).$$

On the tame semistable locus the Swan term vanishes, so

$$f_{A,t} = \tau_t.$$

- (iv) The rational ℓ -adic Tate modules are isomorphic for every $\ell \neq p$ [8, §6, Proposition 6.1 and p. 32], and hence

$$f_{A,t} = f_{B,t}.$$

The component-group estimates of [Theorems 3.1 and 3.5](#) therefore control only the auxiliary component-group and Tamagawa-index defect terms, not a variation of the Artin conductor.

Lemma 4.2 (Prime-to- p isogenies: tame data and Artin conductors). *Let $t \in U$ be as in [Theorem 4.1](#), and let $\varphi : A_{\eta_T} \rightarrow B_{\eta_T}$ be a prime-to-residue-characteristic isogeny whose extension over U is finite étale on identity components. Then:*

- (a) The tame toric contribution is preserved:

$$\tau_t(A) = \tau_t(B).$$

- (b) The rational ℓ -adic Tate modules are isomorphic for every $\ell \neq p$ [8, §6, Proposition 6.1 and p. 32], and hence the Artin conductor exponents agree:

$$f_{A,t} = f_{B,t}.$$

- (c) The component-group map

$$\Phi_{A,t} \rightarrow \Phi_{B,t}$$

has kernel and cokernel annihilated by the isogeny degree. This gives uniform control of Tamagawa-index and local geometric defect terms, but not a conductor-variation inequality.

Proof. Scope. We first separate the two comparison statements that are used in the argument. On the abelian-scheme locus, A_U/U is proper and smooth, so smooth and proper base change applies directly and gives the usual specialization isomorphisms. On the semistable locus for a Néron model, however, A_U is smooth and separated but need not be proper. Thus we do not use smooth and proper base change as a cohomology comparison theorem for the non-proper Néron model itself. Instead, we use the standard Raynaud-extension description of the inertia invariants of the generic-fiber Tate module.

More precisely, for a closed point $t \in U$, pass to the strict henselization of the DVR $\mathcal{O}_{U,t}$, and let the identity component of the special fiber have Raynaud extension

$$0 \rightarrow T_t \rightarrow (A_t)^0 \rightarrow B_t \rightarrow 0,$$

where T_t is a torus of rank τ_t and B_t is an abelian variety ([11, Exposé IX, §3.6]). Then, for $\ell \neq p = \text{char } \kappa(t)$,

$$\dim_{\mathbb{Q}_\ell} V_\ell(A)^{I_t} = 2g - \tau_t.$$

This is a statement about the rational ℓ -adic Tate module of the generic abelian variety and its inertia action. It is not a statement about the full étale cohomology of an arbitrary non-proper Néron model.

For [Item \(i\)](#), on the good-reduction locus A_U/U is an abelian scheme, and smooth and proper base change gives the asserted

specialization isomorphism (cf. [10, Exposé XVI, Théorème 1.2]). On the semistable tame locus, the monodromy formalism [11, Exposé I, §§1–5], together with Raynaud’s semistable reduction theory [11, Exposé IX, §3.6] gives the displayed formula for $V_\ell(A)^{I_t}$, equivalently for

$$H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t}.$$

After shrinking U , the reduction type is constant on the relevant strata, so the function

$$t \mapsto \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t}$$

is constructible on the fixed semistable tame locus.

For [Item \(ii\)](#), let $\varphi_{\eta_T} : A_{\eta_T} \rightarrow B_{\eta_T}$ be one of the prime-to-residue-characteristic isogenies considered on the fixed tame open U . By [Theorem 3.1](#), interpreted in the Dedekind/codimension-one sense fixed in [Remark 2.3](#), it extends to the corresponding morphism of Néron models on the tame locus. The extension and the finite-étale kernel hypothesis are needed for the identity-component and component-group control proved in [Theorems 3.1 and 3.5](#). For the rational cohomological statement itself, the essential input is the classical fact that an isogeny of abelian varieties induces an isomorphism of rational Tate modules:

$$V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\sim} V_\ell(B) := T_\ell(B) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

as $G_{K(t)}$ -representations. Taking I_t -invariants of this isomorphism gives

$$V_\ell(A)^{I_t} \xrightarrow{\sim} V_\ell(B)^{I_t}.$$

Equivalently,

$$H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} \xrightarrow{\sim} H_{\text{ét}}^1(B_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t}.$$

Thus the dimension of the inertia-invariant rational cohomology is preserved under the isogenies considered here. In particular, on the fixed tame semistable locus, the toric-rank contribution appearing in the Raynaud formula is the same for A and B .

For [Item \(iii\)](#), the Artin conductor exponent is computed from the rational ℓ -adic Galois representation

$$H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell) \simeq V_\ell(A)^\vee.$$

Its Artin conductor has the standard tame-plus-wild decomposition

$$f_{A,t} = \text{codim}_{\mathbb{Q}_\ell} H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} + \text{Swan}\left(H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)\right).$$

For semistable reduction, the codimension of the inertia-invariant subspace equals the toric rank τ_t . Therefore

$$f_{A,t} = \tau_t + \text{Swan}\left(H_{\text{ét}}^1(A_{\overline{K(t)}}, \mathbb{Q}_\ell)\right).$$

On the tame semistable locus the Swan term is zero, and hence

$$f_{A,t} = \tau_t.$$

Since φ_{η_T} induces an isomorphism of rational ℓ -adic Galois representations for A and B , their Artin conductor exponents agree:

$$f_{A,t} = f_{B,t}.$$

The component-group estimates proved earlier therefore control only the Tamagawa-index and local geometric defect terms. They should not be read as a conductor-variation inequality. $\square$

Analytically, [Theorem 4.1\(ii\)](#) ensures that the tame pole order of the local L -factor $L_t(A, s)$ remains constant under all prime-to- p isogenies. The component-group estimates above control only the associated Tamagawa-index and local geometric defect terms.

Corollary 4.3 (Invariant local factor on the semistable locus). *Under the hypotheses of [Theorem 4.1](#) with $\ell \neq p$, the invariant local factor is*

$$P_{A,p}^I(X) = \det(1 - X \text{Frob}_p \mid V_\ell(A)^{I_p}).$$

It has degree $2g - \tau_p$ and is invariant under the prime-to- p isogenies considered on U . If the torus in the Raynaud extension is split, then a factor $(1 - p^{-s})^{\tau_p}$ occurs in the corresponding local factor. In the non-split toric case the toric factor is instead determined by Frobenius on the character lattice of the torus.

Example 4.4 (Bad reduction and inertia action). Let $E/\mathbb{Q}$ be an elliptic curve and fix $\ell \neq p$.

Good reduction. If E has good reduction at p , then the special fiber E_p is an abelian variety and I_p acts trivially on $T_\ell(E)$, by the Néron–Ogg–Shafarevich criterion [[5](#), Theorem 1, p. 493]; equivalently,

$$H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} \cong H_{\text{ét}}^1(E_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell),$$

$$\dim_{\mathbb{Q}_\ell} H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 2.$$

The local Euler factor is

$$L_p(E, s)^{-1} = 1 - a_p p^{-s} + p^{1-2s},$$

with $a_p = p + 1 - \#E(\mathbb{F}_p)$ (trace of Frobenius on H^1). Here $\tau_p = 0$ and the tame factor $(1 - p^{-s})^{\tau_p}$ is absent, in agreement with [Corollary 4.3](#).

Split multiplicative reduction. Assume E has split multiplicative reduction at p . Then the identity component of the Néron special fiber is a torus,

$$(E_p)^0 \simeq \mathbb{G}_m, \quad \tau_p = 1.$$

The I_p -action on $H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is unipotent with a single invariant line coming from the toric part, hence

$$\dim_{\mathbb{Q}_\ell} H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 1.$$

The local inverse Euler factor is

$$L_p(E, s)^{-1} = 1 - p^{-s},$$

which is the split toric factor predicted by [Corollary 4.3](#) with $\tau_p = 1$ and has degree $2g - \tau_p = 1$.

Additive potentially good reduction. If E has additive potentially good reduction at p , then wild inertia may act non-trivially on $H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$, and the Swan term can be positive. A tame extension does not, in general, kill wild inertia. After a suitable finite extension, which may be wildly ramified depending on p , the curve may acquire good reduction. This case therefore lies outside the tame semistable mechanism used in [Theorem 4.1](#) and [corollary 4.3](#).

Example 4.5 (Counterexample illustrating necessity of the prime-to- p hypothesis). Let $E/\mathbb{Q}$ admit a 2-isogeny $\varphi_\eta : E \rightarrow E'$ and suppose E has additive reduction at $p = 2$.

Geometric limitation on the special fiber. Extend φ_η to a morphism of Néron models over $\mathbb{Z}_2$. Because the residue characteristic divides the isogeny degree, its finite flat kernel of order 2 need not be étale on the special fiber. Consequently, the finite-étale identity-component mechanism used in the prime-to- p argument is unavailable at this prime.

Let

$$\widehat{\varphi}_\eta : E' \rightarrow E$$

be the dual isogeny. Since

$$\widehat{\varphi}_\eta \circ \varphi_\eta = [2], \quad \varphi_\eta \circ \widehat{\varphi}_\eta = [2],$$

functoriality on component groups shows that the kernel and cokernel of

$$\Phi_{E,2} \rightarrow \Phi_{E',2}$$

are nevertheless annihilated by 2. The excluded prime therefore illustrates the failure of the finite-étale mechanism, not a failure of the component-group annihilation bound.

Galois-theoretic clarification. Although the finite flat kernel may fail to be étale at $p = 2$, this does not imply that the rational ℓ -adic cohomology changes under the isogeny. For every $\ell \neq 2$, the isogeny induces an isomorphism of rational Galois representations

$$V_\ell(E) \xrightarrow{\sim} V_\ell(E'),$$

and hence also an isomorphism on inertia-invariant subspaces

$$H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}_2}, \mathbb{Q}_\ell)^{I_2} \xrightarrow{\sim} H_{\text{ét}}^1(E'_{\overline{\mathbb{Q}}_2}, \mathbb{Q}_\ell)^{I_2}.$$

Thus the failure at $p = 2$ is not a failure of rational Tate-module isogeny invariance. It is only a failure of the tame finite-étale kernel mechanism used to control identity components and component groups.

Local geometric implication. The example shows that outside the prime-to- p range the tame component-group argument no longer applies. One may encounter non-étale finite-flat kernels and wild geometric behavior on the special fiber. This is a failure of the tame geometric control mechanism, not a claim that the Artin conductor of the rational Tate module varies under isogeny.

4.2 Heights and invariant differentials

Definition 4.6 (Uniform control of tame local geometric data). Fix an integer $m \geq 1$ and a dense open U on which m is invertible. Let

$$\varphi : A \longrightarrow B$$

be a degree- m isogeny of abelian varieties whose Néron models are semistable over U , and suppose that the kernel of φ extends to a finite étale subgroup scheme over U .

We say that the tame local geometric data are *uniformly controlled* on U if, for every closed point $t \in U$, the following hold:

- (i) the toric ranks agree:

$$\tau_t(A) = \tau_t(B);$$

- (ii) pullback of invariant differentials along the extended isogeny induces an isomorphism of vector bundles on U :

$$\varphi_\omega^\# : \omega_{B/U} \xrightarrow{\sim} \omega_{A/U}.$$

- (iii) the induced morphism

$$\Phi_{A,t} \longrightarrow \Phi_{B,t}$$

has kernel and cokernel annihilated by m :

$$m \ker(\Phi_{A,t} \rightarrow \Phi_{B,t}) = 0, \quad m \operatorname{coker}(\Phi_{A,t} \rightarrow \Phi_{B,t}) = 0$$

The word “uniformly” means that the same open U and the same integer m work simultaneously for every isogeny in the fixed finite family under consideration. It does not mean that the values of all individually normalized local height functions satisfy an additional point-independent numerical bound.

Proposition 4.7 (Canonical heights and controlled tame local data). *Let*

$$\varphi : A \longrightarrow B$$

be an isogeny satisfying the hypotheses of Definition 4.6, and let L be a symmetric ample line bundle on B . Then

$$\widehat{h}_{\varphi^*L}(P) = \widehat{h}_L(\varphi(P)) \quad \text{for every } P \in A(\mathbb{Q}).$$

On the fixed tame open U , the associated toric ranks, invariant-differential bundles, and component-group defect terms are uniformly controlled precisely in the sense of Definition 4.6. No additional point-independent bound on the values of individually normalized local height functions is asserted.

Proof. The displayed identity is the standard functoriality of the Néron–Tate height under pullback by an isogeny.

On the tame open U , preservation of the toric rank follows from Theorem 4.1(ii). Since the extended isogeny is finite étale over U , its pullback map on invariant differentials is an isomorphism. The kernel and cokernel of

$$\Phi_{A,t} \longrightarrow \Phi_{B,t}$$

are annihilated by m by Theorems 3.1 and 3.5. These are exactly the three conditions in Definition 4.6. $\square$

4.3 Link to moduli-theoretic applications

The toric-rank function $t \mapsto \tau_t$ and the geometric component-order function

$$t \longmapsto \bar{c}_{A,t} = \#\Phi_{A,t}(\overline{\kappa(t)})$$

are constructible and locally constant on the fixed tame open U . The Tamagawa-number function

$$t \longmapsto c_{A,t} = \#\Phi_{A,t}(\kappa(t))$$

is kept separate and is not asserted to be locally constant without control of the Frobenius action. In the next section, these geometric component-group and cohomological controls are applied in two logically distinct settings: first, to explicitly defined Hecke-isogeny endomorphisms of fixed T_ℓ -stable abelian quotients of modular Jacobians; and second, under the stated integral-extension and prime-to-residue-characteristic hypotheses, to genuinely varying abelian varieties related by selected Hecke correspondences on Hodge-type Shimura varieties.

5 Moduli-Theoretic Aspects

We retain the base and notation from Notation 2.1. Theorems 3.1 and 3.5 control Néron models in arithmetic families and under prime-to-residue-characteristic isogenies; Theorem 4.1 describes the corresponding cohomological invariants and conductor behavior. We now treat two logically separate applications: explicitly defined Hecke endomorphisms of fixed modular quotients, and conditional prime-to- p Hecke correspondences on selected Hodge-type Shimura varieties.

Standing convention for the modular-curve application. Let $J = J_0(N)$. For a prime $\ell \nmid N$, write

$$\pi_1, \pi_2 : X_0(N\ell) \longrightarrow X_0(N)$$

for the two standard degeneracy maps and

$$T_\ell = (\pi_1)_* \pi_2^* \in \operatorname{End}_{\mathbb{Q}}(J)$$

for the resulting Hecke endomorphism. The degeneracy maps themselves relate $J_0(N\ell)$ and $J_0(N)$, whereas T_ℓ is an endomorphism of the fixed Jacobian $J_0(N)$. We never identify either degeneracy map with an isogeny between these two Jacobians, and we never decompose T_ℓ into unspecified “morphism factors.”

When applying the prime-to-residue-characteristic isogeny results, we first choose a T_ℓ -stable abelian quotient

$$q : J \twoheadrightarrow A$$

such that T_ℓ descends to an endomorphism

$$u_\ell : A \longrightarrow A$$

which is an isogeny. We then remove all primes dividing

$$N\ell \deg(u_\ell).$$

All component-group conclusions below refer to this explicitly defined isogeny u_ℓ of the fixed abelian variety A .

5.1 Néron models and integral models of modular curves

Notation 5.1 (Integral modular curves and Jacobians). Fix a positive integer $N \geq 1$. Let $\mathcal{X}_0(N)$ denote an integral regular model of the modular curve $X_0(N)$ over $\text{Spec } \mathbb{Z}$ away from a finite set of primes (Deligne–Rapoport and Katz–Mazur; see [16, Introduction, pp. 146–153; Ch. V, pp. 234–257; Ch. VII, pp. 291–313] and [17, Thm. 3.7.1, pp. 104–105; §6.5, p. 164]). Write $J_0(N) = \text{Pic}^0(X_0(N))$ for the Jacobian over $\mathbb{Q}$, and J/S for its Néron model (Definition 2.6 and lemma 2.7). For a prime p , denote by $\Phi_{J,p}$ the component group and by $c_{J,p} = \# \Phi_{J,p}(\mathbb{F}_p)$ its local Tamagawa number (Proposition 2.8).

Remark 5.2 (Degeneracy correspondence and Hecke endomorphism). Let $\ell \nmid N$. Over $\mathbb{Z}[1/N\ell]$, the standard degeneracy maps

$$\pi_1, \pi_2: \mathcal{X}_0(N\ell) \longrightarrow \mathcal{X}_0(N)$$

are finite morphisms representing the two operations of forgetting and quotienting the cyclic subgroup of order ℓ ; see [16, Introduction, pp. 146–153; Ch. V, pp. 234–257; Ch. VII, pp. 291–313] and [17, Thm. 3.7.1, pp. 104–105; §6.5, p. 164]. They induce homomorphisms

$$\pi_2^*: J_0(N) \longrightarrow J_0(N\ell), \quad (\pi_1)_*: J_0(N\ell) \longrightarrow J_0(N),$$

and their composite

$$T_\ell = (\pi_1)_* \circ \pi_2^* \in \text{End}_{\mathbb{Q}}(J_0(N))$$

is the usual Hecke endomorphism.

The Jacobians $J_0(N\ell)$ and $J_0(N)$ generally have different dimensions. Accordingly, neither π_2^* nor $(\pi_1)_*$ is asserted here to be an isogeny between the full Jacobians. Moreover, T_ℓ is not asserted to have degree ℓ^2 . Prime-to- p isogeny arguments will be applied only after specifying a T_ℓ -stable abelian quotient on which the induced endomorphism is an isogeny.

Theorem 5.3 (Hecke endomorphisms on stable isogeny quotients). *Let $N \geq 1$, let $\ell \nmid N$ be prime, and put*

$$J = J_0(N).$$

Let

$$T_\ell = (\pi_1)_* \pi_2^* \in \text{End}_{\mathbb{Q}}(J)$$

be the Hecke endomorphism defined in Remark 5.2. Suppose that

$$q: J \twoheadrightarrow A$$

is an abelian quotient over $\mathbb{Q}$, and that there is an endomorphism $u_\ell \in \text{End}_{\mathbb{Q}}(A)$ satisfying

$$q \circ T_\ell = u_\ell \circ q.$$

Assume that u_ℓ is an isogeny, and set

$$d_\ell = \deg(u_\ell).$$

Define

$$U_{A,\ell} = \text{Spec } \mathbb{Z} \setminus \left(\{ \text{bad-reduction primes of } A \} \cup \{ p : p \mid N\ell d_\ell \} \right).$$

Let $\mathcal{A}$ be the Néron model of A . For every $p \in U_{A,\ell}$, the following statements hold.

- (i) *The isogeny u_ℓ extends uniquely to a morphism*

$$\sqcap_\ell: \mathcal{A}_{\mathbb{Z}_p} \longrightarrow \mathcal{A}_{\mathbb{Z}_p}.$$

Its restriction to the identity component has finite étale kernel of order prime to p .

- (ii) *The induced endomorphism*

$$\bar{u}_{\ell,p}: \Phi_{A,p} \longrightarrow \Phi_{A,p}$$

has kernel and cokernel annihilated by d_ℓ . Consequently, for every prime $r \nmid d_\ell$, it induces an automorphism of the r -primary component of $\Phi_{A,p}$.

- (iii) *For every auxiliary prime $r \neq p$, the map induced by u_ℓ on rational Tate modules,*

$$V_r(u_\ell): V_r(A) \longrightarrow V_r(A),$$

is an automorphism of $G_{\mathbb{Q}}$ -representations. Hence its restriction

$$V_r(A)^{I_p} \longrightarrow V_r(A)^{I_p}$$

is an automorphism commuting with Frobenius.

- (iv) *The inertia-invariant local polynomial*

$$P_{A,p}^I(X) = \det(1 - X \text{Frob}_p \mid V_r(A)^{I_p})$$

and the Artin conductor exponent $f_{A,p}$ are invariants of the fixed representation $V_r(A)$. The Hecke endomorphism u_ℓ acts automorphically on this representation; no variation among different Néron models or different members of a Hecke orbit is asserted.

- (v) *If A has semistable reduction at p , then the toric part of $\mathcal{A}_p^0$ is carried isogenously to itself by $\sqcap_{\ell,p}^0$. In particular, its toric rank*

$$\tau_{A,p} = \text{rank}_{\mathbb{Z}} X_*(T_{A,p})$$

is unchanged by the endomorphism.

Scope of the theorem. The theorem concerns an explicitly defined Hecke endomorphism on one fixed T_ℓ -stable abelian quotient A of $J_0(N)$. Its content is the compatibility of that endomorphism with the Néron model of A and the resulting integral control of the induced component-group endomorphism. The local polynomial and Artin conductor statements are consequences of the fact that an isogeny acts invertibly on rational Tate modules. They are not assertions of variation across a Hecke orbit.

Proof. The relation

$$q \circ T_\ell = u_\ell \circ q$$

defines the Hecke action on the fixed quotient A . Since u_ℓ is an isogeny of degree d_ℓ , the Néron mapping property extends it uniquely over every $\mathbb{Z}_p$. If $p \nmid d_\ell$, its kernel extends, after the stated restriction of the base, as a finite étale subgroup scheme. Therefore the induced morphism on identity components has

finite étale kernel, and [Lemma 3.4](#) and [theorem 3.5](#) shows that the kernel and cokernel of

$$\bar{u}_{\ell,p}: \Phi_{A,p} \longrightarrow \Phi_{A,p}$$

are annihilated by d_ℓ .

For every auxiliary prime r , an isogeny induces an isomorphism on rational Tate modules:

$$V_r(u_\ell): V_r(A) \xrightarrow{\sim} V_r(A).$$

This isomorphism is $G_{\mathbb{Q}}$ -equivariant. It therefore restricts to an automorphism of $V_r(A)^{I_p}$ and commutes with the induced Frobenius action. The statements about the local polynomial and Artin conductor follow immediately. In the semistable case, an isogeny of the identity components induces an isogeny of their maximal tori, so their character and cocharacter groups have the same rank. $\square$

Remark 5.4 (When the quotient hypothesis holds). A standard source of quotients satisfying the hypothesis of [Theorem 5.3](#) is a Hecke-stable quotient attached to a newform. If A_f is the quotient associated with a normalized newform f , then T_ℓ induces an endomorphism determined by the Hecke eigenvalue $a_\ell(f)$. The theorem applies precisely when this induced endomorphism is an isogeny. In the elliptic-curve case, where the induced map is multiplication by an integer $a_\ell(f)$, this requires

$$a_\ell(f) \neq 0,$$

and then

$$\deg([a_\ell(f)]) = a_\ell(f)^2.$$

If $a_\ell(f) = 0$, the induced endomorphism is the zero map and the isogeny conclusions of [Theorem 5.3](#) do not apply.

Corollary 5.5 (Hecke-equivariance of the invariant local factor). *Under the hypotheses of [Theorem 5.3](#), let $p \in U_{A,\ell}$, and choose an auxiliary prime $r \neq p$. Then u_ℓ induces an automorphism*

$$V_r(A)^{I_p} \xrightarrow{\sim} V_r(A)^{I_p}$$

commuting with Frob_p . Consequently,

$$P_{A,p}^I(X) = \det(1 - X \text{Frob}_p | V_r(A)^{I_p})$$

is Hecke-equivariant.

If A has semistable reduction at p , this polynomial has degree

$$\deg P_{A,p}^I = 2 \dim(A) - \tau_{A,p}.$$

If the toric part of the Raynaud extension is split at p , then

$$\begin{aligned} P_{A,p}^I(p^{-s}) &= (1 - p^{-s})^{\tau_{A,p}} Q_{A,p}(p^{-s}), \\ \deg Q_{A,p} &= 2 \dim(A) - 2\tau_{A,p}. \end{aligned}$$

In the non-split case, the toric factor is determined by Frobenius on the character lattice of the torus.

No comparison among different points of a Hecke orbit is asserted.

Proof. Apply [Theorem 5.3](#)(iii)–(v) to the fixed abelian variety A and its isogeny endomorphism u_ℓ . $\square$

Example 5.6 (The Hecke action on $J_0(11)$). Let

$$E = J_0(11): y^2 + y = x^3 - x^2 - 10x - 20.$$

This elliptic curve has conductor 11, so it has good reduction at every prime $p \neq 11$. Hence

$$\tau_{E,p} = 0, \quad \Phi_{E,p} = 0, \quad f_{E,p} = 0 \quad (p \neq 11).$$

For a prime $\ell \neq 11$, the Hecke endomorphism T_ℓ acts on the newform quotient E as multiplication by the integer $a_\ell(E)$:

$$u_\ell = [a_\ell(E)]: E \longrightarrow E.$$

If $a_\ell(E) \neq 0$, this is an isogeny of degree

$$d_\ell = a_\ell(E)^2.$$

Therefore, for every prime

$$p \nmid 11\ell a_\ell(E),$$

the isogeny u_ℓ extends to the Néron model of E , and the induced endomorphism of the component group has kernel and cokernel annihilated by $a_\ell(E)^2$. In this example the component group is already trivial at such p .

For every auxiliary prime $r \neq p$, multiplication by $a_\ell(E)$ is an automorphism of $V_r(E)$. It commutes with Frobenius and inertia, and hence with the good-reduction local polynomial

$$P_{E,p}(X) = 1 - a_p(E)X + pX^2.$$

Equivalently,

$$L_p(E, s)^{-1} = 1 - a_p(E)p^{-s} + p^{1-2s}.$$

If $a_\ell(E) = 0$, the induced Hecke endomorphism is the zero map, not an isogeny. In that case the component-group isogeny conclusions of [Theorem 5.3](#) are not invoked.

Example 5.7 (Boundaries of the Hecke-isogeny framework). The framework of [Theorem 5.3](#) does not apply in any of the following cases:

- (i) $p \mid N$, where the integral modular curve has bad-level geometry;
- (ii) $p = \ell$, where the cyclic- ℓ degeneracy correspondence is not prime to the residue characteristic;
- (iii) $p \mid \deg(u_\ell)$, where the kernel of the induced endomorphism on A need not be étale over $\mathbb{Z}_p$;
- (iv) u_ℓ is not an isogeny, for example when the induced endomorphism on an elliptic newform quotient is $[a_\ell]$ with $a_\ell = 0$.

These exclusions concern the integral finite-étale kernel and component-group argument. They do not alter the general fact that two isogenous abelian varieties have isomorphic rational Tate-module representations.

For every T_ℓ -stable quotient A satisfying [Theorem 5.3](#), the Hecke endomorphism u_ℓ acts automorphically on $V_r(A)$ and on $V_r(A)^{I_p}$ at the permitted primes. The associated component-group endomorphism has d_ℓ -primary kernel and cokernel. These are endomorphism statements for the fixed Néron model of A , not orbitwise comparison statements among different Jacobians.

5.2 Relation with Shimura varieties

For ordinary Hodge-type Shimura varieties at hyperspecial level, van Hoften proves that the prime-to- p Hecke orbit of every ordinary point is Zariski dense in the special fibre [18, Theorem I, p. 848]. Our use below is narrower: we do not prove orbit-density, but only record conditional tame Néron-model uniformity along those prime-to-residue-characteristic Hecke correspondences that extend integrally with finite étale kernels.

More generally, D'Addezio and van Hoften prove the Hecke orbit conjecture for Shimura varieties of Hodge type at odd primes of good reduction: a non-empty reduced closed subvariety of a central leaf stable under prime-to- p Hecke operators equals the whole central leaf [19, Theorem I (= Theorem 8.3.2), p. 2].

Notation 5.8 (Integral models and universal abelian schemes). Let (G, X) be a Shimura datum of Hodge type with reflex field E . Fix a compact open subgroup

$$K = K^p K_p \subseteq G(\mathbb{A}_f)$$

such that K_p is hyperspecial. Let $v \mid p$ be a finite place of E at which the chosen Hodge-type datum has good hyperspecial reduction, and write

$$B_v := \operatorname{Spec} \mathcal{O}_{E,(v)}.$$

Let

$$\pi_{\mathcal{S}}: \mathcal{S}_K \longrightarrow B_v$$

denote the corresponding integral canonical model. When a fine moduli interpretation is available, let

$$\pi_{\mathcal{A}}: \mathcal{A} \longrightarrow \mathcal{S}_K$$

denote the universal abelian scheme of relative dimension g . In the general Hodge-type case, $\mathcal{A}$ denotes the abelian scheme obtained from the fixed Hodge embedding after passing, if necessary, to a sufficiently small neat level at which the universal family exists.

For every B_v -scheme T and every T -valued point

$$x: T \longrightarrow \mathcal{S}_K,$$

write

$$\mathcal{A}_x := \mathcal{A} \times_{\mathcal{S}_K, x} T \longrightarrow T$$

for the pullback abelian scheme.

When compactifications are needed, we use a smooth toroidal compactification of $\mathcal{S}_K$ with normal-crossings boundary as in [20, Theorem 2, p. 396]; the corresponding minimal compactification is the projective normal compactification of [20, Theorem 3, p. 397]. No extension of the universal abelian scheme across the boundary is assumed unless it is stated explicitly.

Remark 5.9 (Prime-to- p Hecke correspondences). Let $h \in G(\mathbb{A}_f^p)$. Put

$$K_h := K \cap hKh^{-1}.$$

On the generic fibres, the Hecke correspondence determined by h is represented by the diagram

$$\begin{array}{ccc} \operatorname{Sh}_{K_h}(G, X) & \xrightarrow{p_1} & \operatorname{Sh}_K(G, X), \\ \operatorname{Sh}_{K_h}(G, X) & \xrightarrow{p_2} & \operatorname{Sh}_K(G, X). \end{array}$$

where the two morphisms are induced by the inclusion of levels and by right translation by h , respectively.

We use this correspondence integrally only on an explicitly specified dense open

$$U_h \subseteq B_v$$

on which the two projections extend to finite morphisms between the corresponding integral models and the induced morphism between the pulled-back universal abelian schemes is an isogeny of degree invertible on U_h . In particular, its kernel is finite étale over U_h .

No assertion is made that every prime-to- p Hecke correspondence extends over the whole integral model with these properties.

Theorem 5.10 (Conditional same-place Hecke comparison for Hodge-type integral models). *Retain the notation of Notation 5.8. Let*

$$U_h \subseteq B_v$$

be a dense open over which the Hecke correspondence determined by $h \in G(\mathbb{A}_f^p)$ extends integrally and induces prime-to-residue-characteristic isogenies between the relevant pullbacks of the universal abelian scheme.

Let R be a henselian discrete valuation ring equipped with a morphism

$$b: \operatorname{Spec} R \longrightarrow U_h.$$

Write $K_R = \operatorname{Frac}(R)$, and assume that the residue field k of R is a finite extension of the residue field of the image of the closed point of $\operatorname{Spec} R$ in U_h . Put

$$p_R := \operatorname{char}(k).$$

Thus k is finite, and its arithmetic Frobenius

$$\operatorname{Frob}_k \in G_{K_R}/I_{K_R}$$

is well defined.

Suppose that

$$x, x': \operatorname{Spec} R \longrightarrow \mathcal{S}_K$$

are morphisms over U_h , meaning that

$$\pi_{\mathcal{S}} \circ x = b = \pi_{\mathcal{S}} \circ x'.$$

Assume that x and x' are related by the integral Hecke correspondence and that it induces an R -isogeny

$$\varphi_h: \mathcal{A}_x \longrightarrow \mathcal{A}_{x'}$$

of degree d_h , with

$$p_R \nmid d_h.$$

Then:

- (i) The kernel of φ_h is a finite étale group scheme over R . Since $\mathcal{A}_x$ and $\mathcal{A}_{x'}$ are abelian schemes over R , they are the Néron models of their generic fibres. Their special fibres are connected, and hence

$$\Phi_{\mathcal{A}_x, k} = 0, \quad \Phi_{\mathcal{A}_{x'}, k} = 0.$$

Thus the induced map on component groups is the unique isomorphism

$$0 \longrightarrow 0.$$

- (ii) For every prime $\ell \neq p_R$, the generic-fibre isogeny induces an isomorphism of G_{K_R} -representations

$$H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell).$$

Both representations are unramified because both abelian varieties have good reduction over R . Consequently,

$$H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell)^{I_{K_R}} = H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell)^{I_{K_R}}$$

and similarly for $\mathcal{A}_{x'}$. The above isomorphism commutes with the common Frobenius operator Frob_k .

- (iii) The two Artin conductor exponents vanish:

$$f_{K_R}(\mathcal{A}_x) = f_{K_R}(\mathcal{A}_{x'}) = 0.$$

Their toric ranks also vanish:

$$\tau_x = \tau_{x'} = 0.$$

- (iv) The corresponding good-reduction local polynomials agree:

$$\begin{aligned} \det(1 - X\text{Frob}_k \mid H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell)) \\ = \det(1 - X\text{Frob}_k \mid H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell)). \end{aligned}$$

These conclusions compare two moduli points lying over the same morphism

$$\text{Spec } R \longrightarrow U_h$$

and an isogeny defined over the same DVR R . No comparison is asserted between points lying over different arithmetic places, between unrelated residue fields, or across boundary points at which the universal family is no longer an abelian scheme.

Scope and relation to the global theory. The theorem is a same-base and same-place comparison for pullbacks of the universal abelian scheme. The arithmetic morphism

$$\text{Spec } R \longrightarrow U_h$$

specifies the arithmetic place, while

$$x, x': \text{Spec } R \longrightarrow \mathcal{S}_K$$

specify the two moduli points lying over that same arithmetic base point.

Because these pullbacks are abelian schemes over R , their reduction is good and their Néron component groups, toric

ranks, and Artin conductor exponents are trivial. The substantive comparison is therefore the Frobenius-equivariant isomorphism of their rational ℓ -adic cohomology induced by the specified Hecke isogeny.

No semistable-boundary or degenerating-family conclusion is included in this theorem.

Proof. By definition,

$$\mathcal{A}_x = \mathcal{A} \times_{\mathcal{S}_{K,x}} \text{Spec } R, \quad \mathcal{A}_{x'} = \mathcal{A} \times_{\mathcal{S}_{K,x'}} \text{Spec } R.$$

Since $\mathcal{A} \rightarrow \mathcal{S}_K$ is an abelian scheme, both $\mathcal{A}_x \rightarrow \text{Spec } R$ and $\mathcal{A}_{x'} \rightarrow \text{Spec } R$ are abelian schemes.

The degree d_h is invertible in R . Therefore the kernel of the R -isogeny

$$\varphi_h: \mathcal{A}_x \longrightarrow \mathcal{A}_{x'}$$

is finite étale over R . An abelian scheme over a DVR is the Néron model of its generic fibre, and its special fibre is geometrically connected. Hence both component groups are trivial:

$$\Phi_{\mathcal{A}_x, k} = \Phi_{\mathcal{A}_{x'}, k} = 0.$$

For every $\ell \neq p_R$, an isogeny induces an isomorphism on rational ℓ -adic Tate modules and therefore an isomorphism

$$H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell)$$

of G_{K_R} -representations.

Because both abelian varieties have good reduction over R , the Néron–Ogg–Shafarevich criterion shows that these representations are unramified. Their inertia-invariant subspaces are therefore the full cohomology spaces, their Swan conductors and Artin conductor exponents vanish, and their toric ranks are zero.

Finally, the G_{K_R} -equivariance of the cohomological isomorphism implies equivariance for the common arithmetic Frobenius Frob_k . Taking characteristic polynomials gives the equality in (iv). $\square$

Corollary 5.11 (Same-place invariance of good-reduction local factors under a Hecke isogeny). *Under the hypotheses and notation of Theorem 5.10, define*

$$P_{\mathcal{A}_x/K_R}(X) := \det(1 - X\text{Frob}_k \mid H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell))$$

and

$$P_{\mathcal{A}_{x'}/K_R}(X) := \det(1 - X\text{Frob}_k \mid H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell)).$$

Then

$$P_{\mathcal{A}_x/K_R}(X) = P_{\mathcal{A}_{x'}/K_R}(X).$$

Both polynomials have degree $2g$. Equivalently,

$$\deg P_{\mathcal{A}_x/K_R} = \deg P_{\mathcal{A}_{x'}/K_R} = 2g.$$

Since both abelian varieties have good reduction over R , their inertia actions are trivial, so that

$$H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell)^{I_{K_R}} = H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell)^{I_{K_R}}$$

and

$$H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell)^{I_{K_R}} = H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell).$$

Consequently, the same equality may be written in inertia-invariant form:

$$\begin{aligned} \det(1 - X\text{Frob}_k \mid H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell)^{I_{K_R}}) \\ = \det(1 - X\text{Frob}_k \mid H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell)^{I_{K_R}}). \end{aligned}$$

No equality is asserted between local polynomials attached to different arithmetic places or unrelated residue fields.

Proof. By [Theorem 5.10\(ii\)](#), the Hecke isogeny induces a G_{K_R} -equivariant isomorphism

$$H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell).$$

It therefore commutes with the common arithmetic Frobenius Frob_k , proving equality of the characteristic polynomials.

Both cohomology spaces have dimension $2g$. Their inertia actions are trivial because $\mathcal{A}_x$ and $\mathcal{A}_{x'}$ are abelian schemes over R , hence have good reduction. This proves all assertions. $\square$

Example 5.12 (Siegel modular case). Take $G = \text{GSp}_{2g}$ with hyperspecial K_p ; then $\mathcal{S}_K$ is the Siegel moduli scheme parametrizing principally polarized abelian schemes $\mathcal{A} \rightarrow \mathcal{S}_K$ of relative dimension g ([3, Ch. IV, §7, pp. 132–136]). For every rational prime $\ell \neq p$, the ℓ -adic Tate module $T_\ell(\mathcal{A})$ is equipped with the standard symplectic form preserved by G , and the prime-to- p Hecke algebra acts, on the stated integral locus, through correspondences induced by prime-to- p isogenies preserving this symplectic structure up to the usual similitude factor.

Geometric picture. Let

$$b: \text{Spec } R \longrightarrow U_h$$

be a morphism from a henselian discrete valuation ring R , with residue field k , fraction field K_R , and $p_R = \text{char}(k)$. Let

$$x, x': \text{Spec } R \longrightarrow \mathcal{S}_K$$

be two morphisms over U_h related by the extended Hecke correspondence. Assume that the induced isogeny

$$\varphi_h: (\mathcal{A}_x, \lambda_x) \longrightarrow (\mathcal{A}_{x'}, \lambda_{x'})$$

is defined over R and has degree prime to p_R .

Both $\mathcal{A}_x$ and $\mathcal{A}_{x'}$ are abelian schemes over R . Consequently, both have good reduction, their special fibres are connected, and

$$\Phi_{\mathcal{A}_x, k} = \Phi_{\mathcal{A}_{x'}, k} = 0.$$

The induced map on component groups is therefore

$$0 \longrightarrow 0.$$

Thus no nontrivial component-group or Tamagawa-index comparison arises for these two pullback abelian schemes.

Cohomological and arithmetic consequences. For every $\ell \neq p_R$, the isogeny over the common local field K_R induces a G_{K_R} -equivariant isomorphism

$$H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell).$$

Because both abelian varieties have good reduction, inertia acts trivially and hence

$$H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell)^{I_{K_R}} = H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell),$$

with the analogous equality for $\mathcal{A}_{x'}$.

The induced isomorphism is Frob_k -equivariant. Moreover,

$$\tau_x = \tau_{x'} = 0$$

and

$$f_{K_R}(\mathcal{A}_x) = f_{K_R}(\mathcal{A}_{x'}) = 0.$$

The local polynomials therefore satisfy

$$\begin{aligned} \det(1 - X\text{Frob}_k \mid H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{K_R}}, \mathbb{Q}_\ell)) \\ = \det(1 - X\text{Frob}_k \mid H_{\text{ét}}^1((\mathcal{A}_{x'})_{\overline{K_R}}, \mathbb{Q}_\ell)). \end{aligned}$$

Analytic viewpoint. The two good-reduction local factors coincide because the Hecke isogeny identifies the corresponding representations of the same local Galois group and commutes with the same Frobenius operator. This is a pairwise comparison over the fixed arithmetic morphism

$$\text{Spec } R \longrightarrow U_h.$$

It gives no comparison between points lying over different arithmetic places or having unrelated residue fields.

Example 5.13 (Scope boundary at parahoric or residue-characteristic level). Suppose that K_p is parahoric rather than hyperspecial, or that the degree of the Hecke isogeny is divisible by the residue characteristic. These situations lie outside the hypotheses of [Theorem 5.10](#).

At parahoric level, the integral model may be singular, and an extension of the relevant universal family with the abelian-scheme properties required in [Theorem 5.10](#) is not assumed. Likewise, if

$$p_R \mid \deg(\varphi_h),$$

the kernel of an integral Hecke isogeny need not be finite étale over R .

Accordingly, [Theorem 5.10](#) makes no component-group, good-reduction, or local-factor assertion at such points. Any comparison there requires a separate analysis of the relevant local model, the extension of the universal family, and the integral kernel of the correspondence.

This exclusion does not alter the general fact that an actual isogeny of generic fibres over one common local field induces an isomorphism of rational ℓ -adic Tate modules for $\ell \neq p_R$. Hence the rational Galois representations and their Artin conductor exponents remain invariant under such an isogeny whenever the comparison is made over the same local field. What may fail is the finite-étale integral-kernel mechanism required by the theorem.

Under the same-base and same-place hypotheses of [Theorem 5.10](#), [Corollary 5.11](#) gives equality of the good-reduction local factors for the two pullback abelian schemes connected by the specified Hecke isogeny. Their Artin conductor exponents, toric ranks, and Néron component groups are all trivial:

$$f_{K_R}(\mathcal{A}_x) = f_{K_R}(\mathcal{A}_{x'}) = 0, \quad \tau_x = \tau_{x'} = 0,$$

and

$$\Phi_{\mathcal{A}_x, k} = \Phi_{\mathcal{A}_{x'}, k} = 0.$$

No semistable-boundary, component-group-variation, or orbit-wise global L -function statement is deduced from the Hodge-type comparison.

5.3 Forward linkage to arithmetic applications

[Theorem 5.3](#) supplies the fixed-quotient modular input for component-group control under Hecke endomorphisms. Separately, [Theorem 5.10](#) supplies a same-place good-reduction comparison for two pullbacks of the universal abelian scheme on the Hodge-type integral model.

For the Hodge-type application, equality of local factors is used only when both moduli points lie over one common henselian discrete valuation ring, with one common residue field, one common local Galois group, and one common Frobenius operator. No comparison is made between Frobenius characteristic polynomials attached to different arithmetic places.

6 Applications to Number Theory

We keep the base and notation from [Notation 2.1](#). Structural controls for Néron models in families and under prime-to-residue-characteristic isogenies were established in [Theorems 3.1](#) and [3.5](#). The cohomological input is the preservation of inertia-invariant $H_{\text{ét}}^1$ on the fixed tame locus, as in [Theorem 4.1](#). The applications below concern tame local Euler-factor shapes, component-group defect terms, and compatibility of local height decompositions. They do not assert conductor variation under isogeny and do not prove a new uniform positive height gap.

Standing hypotheses for [Section 6](#). All local and global bounds below are invoked only on the dense tame locus U from [Theorem 3.5](#); in particular, every prime-to- p isogeny/Hecke map is used only where its kernel is finite étale and the induced morphism on component groups has uniformly bounded kernel/cokernel ([Theorems 3.1](#) and [3.5](#)). Any modular-curve Hecke input is taken from [Theorem 5.3](#) only through an explicitly defined isogeny endomorphism u_ℓ of a fixed T_ℓ -stable quotient. The Hodge-type correspondence statements remain subject to the separate hypotheses of [Theorem 5.10](#).

6.1 Zeta functions and L -functions

Proposition 6.1 (Rational L -factor invariance and separate Tamagawa control). *Let $A/\mathbb{Q}$ and $B/\mathbb{Q}$ be isogenous abelian varieties. Then, for every prime p and every auxiliary $\ell \neq p$,*

the rational ℓ -adic representations $V_\ell(A)$ and $V_\ell(B)$ are isomorphic. Consequently the local Euler factors and Artin conductor exponents agree:

$$L_p(A, s) = L_p(B, s), \quad f_{A,p} = f_{B,p}.$$

On the fixed tame prime-to- p locus of [Theorem 3.5](#), the additional Néron-model component-group maps have kernel and cokernel annihilated by a bounded power of the isogeny degree. These bounds control Tamagawa-index and local geometric defect terms only; they do not measure variation of local L -factors or Artin conductors.

Proof. An isogeny of abelian varieties induces an isomorphism of rational Tate modules

$$V_\ell(A) \xrightarrow{\sim} V_\ell(B)$$

as Galois representations. Hence the characteristic polynomial of Frobenius on inertia invariants, and the Artin conductor of the representation, are the same for A and B . The component-group estimates from [Theorems 3.1](#) and [3.5](#) are separate integral Néron-model estimates and control only Tamagawa-index and auxiliary local geometric defect terms. $\square$

Corollary 6.2 (Local conductor stability). *Under the hypotheses of [Proposition 6.1](#), the local Artin conductor exponents of A and B agree at every prime where the comparison is made:*

$$f_{B,p} = f_{A,p}.$$

The component-group estimates used above control only Tamagawa-index and local geometric defect terms; they do not give, and are not used to give, a conductor-variation inequality.

Proof. The rational ℓ -adic Tate modules of isogenous abelian varieties are isomorphic as Galois representations. Hence their Artin conductor exponents agree. The component-group bounds from [Theorems 3.1](#) and [3.5](#) are separate Tamagawa-index estimates and do not measure variation of the Artin conductor. $\square$

Example 6.3 (Hecke endomorphisms on modular quotients). Let A be a T_ℓ -stable abelian quotient of $J_0(N)$, and assume that the induced endomorphism

$$u_\ell: A \longrightarrow A$$

is an isogeny of degree d_ℓ . For

$$p \nmid N d_\ell$$

and every auxiliary prime $r \neq p$, the induced map

$$V_r(u_\ell): V_r(A) \xrightarrow{\sim} V_r(A)$$

is a $G_{\mathbb{Q}}$ -equivariant automorphism. Hence it preserves the inertia-invariant subspace and commutes with Frobenius:

$$V_r(A)^{I_p} \xrightarrow{\sim} V_r(A)^{I_p}.$$

The local polynomial

$$P_{A,p}^I(X) = \det(1 - X \text{Frob}_p \mid V_r(A)^{I_p})$$

and the Artin conductor exponent $f_{A,p}$ are therefore compatible with the Hecke action.

Separately, the induced endomorphism

$$\Phi_{A,p} \longrightarrow \Phi_{A,p}$$

has kernel and cokernel annihilated by d_ℓ . This is an integral component-group statement for the fixed abelian variety A ; it is not a comparison with an unspecified Jacobian J' .

Example 6.4 (Necessity of the prime-to- p condition for geometric component control). Let $E/\mathbb{Q}$ admit a p -isogeny with additive reduction at p . At this prime the kernel of the isogeny need not extend as a finite étale subgroup scheme of the Néron model. Hence the prime-to- p mechanism used in [Theorems 3.1](#) and [3.5](#) to control identity components and component groups does not apply.

This failure is purely geometric. It does not mean that rational ℓ -adic inertia invariants, local L -factors, or Artin conductor exponents change under the isogeny. For every $\ell \neq p$, the isogeny still induces an isomorphism of rational Galois representations

$$V_\ell(E) \xrightarrow{\sim} V_\ell(E'),$$

and therefore the inertia-invariant subspaces and Artin conductor exponents agree. Thus the excluded prime illustrates only the failure of finite-étale kernel and component-group control, not a failure of rational ℓ -adic or conductor invariance.

By [Proposition 6.1](#) and [Corollary 6.2](#), isogenies and prime-to- p Hecke correspondences preserve the rational local L -factors and Artin conductor exponents at the primes where the comparison is made. The separate Néron-model estimates from [Theorems 3.1](#) and [3.5](#) control Tamagawa-index and local geometric defect terms only.

6.2 Height theory and rational points

Proposition 6.5 (Uniform tame geometric control under isogeny). *Let $A/\mathbb{Q}$ be an abelian variety. Fix a finite family of isogenies*

$$\varphi_i : A_i \longrightarrow B_i$$

whose degrees divide a fixed integer m . Then there exists a finite set of primes Σ , containing all primes dividing m , such that on

$$U = \text{Spec } \mathbb{Z} \setminus \Sigma$$

every φ_i satisfies the conditions of [Definition 4.6](#).

Moreover, for every symmetric ample line bundle L_i on B_i ,

$$\widehat{h}_{\varphi_i^* L_i}(P) = \widehat{h}_{L_i}(\varphi_i(P)) \quad \text{for every } P \in A_i(\mathbb{Q}).$$

Thus “uniform control” in this proposition refers exactly to the common-open toric-rank, invariant-differential, and component-group statements of [Definition 4.6](#). It does not assert a new uniform bound for the numerical values of individual local height functions.

Proof. Because the family is finite, one may enlarge Σ so that all the relevant Néron models are semistable over U , every isogeny degree is invertible on U , and every isogeny kernel extends to a finite étale subgroup scheme over U .

The toric-rank equality, invariant-differential isomorphism, and component-group annihilation statements then follow from [Theorems 3.1](#), [3.5](#) and [4.1](#). The canonical height identity is the standard functoriality of the Néron–Tate height under isogeny. $\square$

Corollary 6.6 (Finiteness criterion for rational points). *Let $C/\mathbb{Q}$ be a smooth projective curve and let $\phi : C \rightarrow A$ be a nonconstant morphism to an abelian variety $A/\mathbb{Q}$ with Néron model A/S . Assume:*

- (a) $A(\mathbb{Q})$ has rank 0;
- (b) the map ϕ has finite fibers on $C(\mathbb{Q})$.

Then $C(\mathbb{Q})$ is finite. (Compare the general Mordell–Lang/Faltings finiteness perspective in [\[12\]](#).) Moreover, if A varies within a prime-to- Σ isogeny class for some finite Σ , and (a)–(b) hold for one representative, then they hold uniformly across the class and the finiteness of $C(\mathbb{Q})$ persists.

Proof. By (a), $A(\mathbb{Q})$ is finite. Thus $\phi(C(\mathbb{Q}))$ is finite. By (b), each rational point in the image has finitely many preimages, hence $C(\mathbb{Q})$ is finite. For the final statement, use the standard invariance of the Mordell–Weil rank under isogeny. Moreover, if $\psi : A \rightarrow B$ is an isogeny, then replacing $\phi : C \rightarrow A$ by $\psi \circ \phi : C \rightarrow B$ preserves the finite-fiber condition on $C(\mathbb{Q})$. $\square$

Example 6.7 (Application to modular curves via rank zero). Let X be a modular curve (e.g. a quotient of $X_0(N)$) with Jacobian J and Abel–Jacobi map $\phi : X \rightarrow J$. If $J(\mathbb{Q})$ has rank 0 (this occurs for several small levels), then [Corollary 6.6](#) gives $X(\mathbb{Q})$ finite. The Hecke-endomorphism statement of [Theorem 5.3](#) does not produce additional curves or imply persistence of this finiteness conclusion across a Hecke orbit.

Example 6.8 (Necessity of the rank hypothesis). Let $E/\mathbb{Q}$ be an elliptic curve of positive rank and take $C = E$ with $\phi = \text{id}_E$. Then $C(\mathbb{Q})$ is infinite, so [Corollary 6.6](#) fails without (a).

[Proposition 6.5](#) records the standard functoriality of canonical heights together with the precisely defined tame geometric control of [Definition 4.6](#). Independently, under the rank-0 and finite-fiber hypotheses of [Corollary 6.6](#), one obtains the stated finiteness conclusion for $C(\mathbb{Q})$.

6.3 Link to examples

The fixed-isogeny control of tame local factors and component-group data, together with standard canonical-height functoriality, provides the framework for the examples in [Section 7](#).

7 Examples and Computations

We illustrate the structural and cohomological statements from [Theorems 3.1](#), [3.5](#) and [4.1](#) and the arithmetic consequences

from [Corollaries 4.3, 6.2 and 6.6](#) and [propositions 6.1 and 6.5](#) by working out two standard testbeds: an elliptic curve with CM and a modular Jacobian. Throughout we keep the notation from [Notation 2.1](#), [definitions 2.6 and 2.13](#), and [construction 2.11](#).

Example 7.1 (The curve $y^2 = x^3 - x$: tame isogeny control, component groups, and local factors). Let

$$E/\mathbb{Q} : y^2 = x^3 - x.$$

This curve has $j(E) = 1728$, admits the 2-isogeny

$$\varphi_\eta : E \longrightarrow E', \quad \varphi_\eta(x, y) = \left(\frac{x^2 - 1}{x}, \frac{y(x^2 + 1)}{x^2} \right),$$

where

$$E' : y^2 = x^3 + 4x.$$

The discriminants are

$$\Delta_E = 64, \quad \Delta_{E'} = -4096.$$

Set

$$T = S = \operatorname{Spec} \mathbb{Z}, \quad U = T^\circ = \operatorname{Spec} \mathbb{Z} \setminus \{2\}.$$

Then the isogeny degree is invertible on U , and the kernel of φ_η extends over U to a finite étale subgroup of $E[2]$. Hence the prime-to-residue-characteristic conclusions of [Theorem 3.1](#) and [Theorem 3.5](#) apply on U .

Good-reduction and component-group computation on U . For every odd prime $p \neq 2$, both E and E' have good reduction. Thus

$$\Phi_{E,p} = \Phi_{E',p} = 0, \quad c_{E,p} = c_{E',p} = 1.$$

The induced morphism on component groups

$$\Phi_{E,p} \longrightarrow \Phi_{E',p}$$

has kernel and cokernel killed by 2, trivially in this good-reduction case. Therefore, for every rational prime $q \neq 2$, the function

$$p \longmapsto v_q(\#\Phi_{E,p})$$

is constant on U , hence upper semi-continuous. This gives a concrete instance of the fixed-tame-open control asserted in [Theorem 3.5](#).

General multiplicative comparison. More generally, if $F/\mathbb{Q}$ is an elliptic curve admitting a 2-isogeny $F \rightarrow F'$, and $p \neq 2$ is a split multiplicative prime lying in the chosen tame locus, then

$$F_p^0 \simeq \mathbb{G}_m, \quad \Phi_{F,p} \simeq \mathbb{Z}/\operatorname{ord}_p(\Delta_F)\mathbb{Z}.$$

The extended isogeny is finite étale on identity components, and

$$\ker(\Phi_{F,p} \rightarrow \Phi_{F',p}), \quad \operatorname{coker}(\Phi_{F,p} \rightarrow \Phi_{F',p})$$

are killed by 2. Consequently the associated Tamagawa-index defect is controlled by the isogeny degree; in the elliptic 2-isogeny case one obtains only 2-power variation in the component-group quotient.

Local factors and inertia on the good-reduction locus. For $p \neq 2$, the local Euler factor of E has the usual good reduction form

$$L_p(E, s)^{-1} = 1 - a_p p^{-s} + p^{1-2s},$$

and for $\ell \neq p$,

$$\dim_{\mathbb{Q}_\ell} H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 2.$$

Since E and E' are isogenous, their rational ℓ -adic Tate modules are isomorphic. Thus the Artin conductor exponents agree:

$$f_{E,p} = f_{E',p}.$$

On U , in fact,

$$\frac{L_p(E', s)}{L_p(E, s)} = 1, \quad f_{E',p} - f_{E,p} = 0.$$

This is the cohomological invariance predicted by [Theorem 4.1](#), not a new conductor-variation estimate.

The excluded prime $p = 2$. At $p = 2$, Tate's algorithm gives additive reduction for E (of Kodaira type III). Hence 2 is excluded from the tame open U . The 2-isogeny is no longer prime to the residue characteristic, and its kernel need not extend as a finite étale subgroup scheme over $\mathbb{Z}_2$. Thus the prime-to- p conclusions of [Theorem 3.1](#) and [Theorem 3.5](#) do not apply at 2.

Invariant differentials and canonical heights. Let $\mathcal{E}/S$ be the Néron model of E , and let

$$\omega = \frac{dx}{2y}$$

be its invariant differential. On the tame locus, pullback of invariant differentials under the extended isogeny satisfies the condition in [Definition 4.6](#). Independently, the canonical heights satisfy

$$\widehat{h}_{\varphi^*L}(P) = \widehat{h}_L(\varphi(P)).$$

No separate numerical bound for an individual local-height contribution is asserted.

Example 7.2 (Jacobians of modular curves). We detail the case $J_0(11)$, which is an elliptic curve $E_{11}/\mathbb{Q}$ (the Jacobian of $X_0(11)$), and then indicate the general pattern.

Local structure at $p = 11$. On the regular integral model of $X_0(11)$ (Deligne–Rapoport; Katz–Mazur), the Néron model of $J_0(11)$ over $\mathbb{Z}$ has split multiplicative reduction at $p = 11$ of type I_5 . Hence

$$\tau_{11} = 1, \quad c_{11} = 5, \quad f_{11} = 1,$$

and for $p \neq 11$ one has good reduction with $c_p = 1$ and $f_p = 0$. This fits the template from [Corollary 4.3](#):

$$L_{11}(J_0(11), s)^{-1} = 1 - 11^{-s}.$$

Indeed, $g = 1$ and $\tau_{11} = 1$, so the invariant local polynomial has degree $2g - \tau_{11} = 1$.

Hecke action on the elliptic newform quotient. Let $\ell \neq 11$ be prime. The Hecke endomorphism T_ℓ acts on

$$J_0(11) = E_{11}$$

as multiplication by the Fourier coefficient $a_\ell(E_{11})$:

$$u_\ell = [a_\ell(E_{11})].$$

If $a_\ell(E_{11}) \neq 0$, then u_ℓ is an isogeny of degree

$$d_\ell = a_\ell(E_{11})^2.$$

For every prime

$$p \nmid 11\ell a_\ell(E_{11}),$$

it extends to the Néron model and induces

$$\Phi_{E_{11},p} \longrightarrow \Phi_{E_{11},p}$$

with kernel and cokernel annihilated by $a_\ell(E_{11})^2$. For every auxiliary prime $r \neq p$, it also induces an automorphism

$$V_r(E_{11})^{I_p} \xrightarrow{\sim} V_r(E_{11})^{I_p}$$

commuting with Frobenius. Since E_{11} has good reduction at every $p \neq 11$, one has

$$\tau_{E_{11},p} = 0, \quad \Phi_{E_{11},p} = 0, \quad f_{E_{11},p} = 0.$$

If $a_\ell(E_{11}) = 0$, the Hecke endomorphism is not an isogeny and this prime-to- p isogeny argument is not applied.

Finiteness and the role of the Hecke action. Since $J_0(11)$ has Mordell–Weil rank 0 over $\mathbb{Q}$, and the Abel–Jacobi map

$$X_0(11) \longrightarrow J_0(11)$$

has finite fibres on rational points, [Corollary 6.6](#) implies that $X_0(11)(\mathbb{Q})$ is finite. This conclusion does not depend on constructing a family of Hecke-related Néron models.

For any T_ℓ -stable rank-zero quotient A of $J_0(N)$ on which T_ℓ induces an isogeny u_ℓ , the conclusions of [Theorem 5.3](#) describe the action of u_ℓ on the fixed Néron model, component groups, and rational Tate modules of A . No persistence of the finiteness conclusion along an undefined collection of “Hecke translates” is asserted.

Example 7.3 (Tate curve with split multiplicative reduction). Fix a prime p and take a parameter $q \in p\mathbb{Z}_p$ with $n := v_p(q) \geq 1$. The Tate curve $E_q/\mathbb{Q}_p$ satisfies

$$E_q(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times / q^\mathbb{Z}, \quad \text{Kodaira type } I_n, \quad \tau_p = 1,$$

and the Néron special fiber has $(E_q)_p^0 \simeq \mathbb{G}_m$ and $\Phi_{E_q,p} \simeq \mathbb{Z}/n\mathbb{Z}$. Hence by [Proposition 2.8](#) the Tamagawa number is $c_p = n$, and by [Corollary 4.3](#)

$$L_p(E_q, s)^{-1} = 1 - p^{-s}.$$

Here the toric rank $\tau_p = 1$ accounts for the full degree-1 invariant local factor, since $2g - \tau_p = 1$; the explicit $\#\Phi_{E_q,p} = n$ realises the template of [Corollary 4.3](#) and [Proposition 2.8](#).

Example 7.4 (Potentially good, wild at p). Let $E/\mathbb{Q}$ be an elliptic curve with additive, potentially good reduction at a prime p (e.g. CM $j = 0$ or 1728 at certain p). Over the original local field $\mathbb{Q}_p$, this reduction is not semistable, so the Raynaud semistable formula

$$\text{codim}_{\mathbb{Q}_\ell} H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)^{I_p} = \tau_p$$

does not apply.

The Artin conductor must instead be computed from the general formula

$$f_{E,p} = \text{codim}_{\mathbb{Q}_\ell} H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)^{I_p} + \text{Swan}\left(H_{\text{ét}}^1(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)\right).$$

In particular, wild inertia may contribute a positive Swan term. After a suitable finite extension, possibly wildly ramified, E may acquire good reduction.

Thus this example lies outside the tame semistable mechanism of [Theorem 4.1](#) and [corollary 4.3](#); it illustrates why the general conductor formula of [Definition 2.13](#) must be retained outside that locus.

Example 7.5 (Product surface with mixed reduction). Let $A = E_1 \times E_2$ over $\mathbb{Q}$ with Néron model $\mathcal{A}/S$. Suppose at a prime p that E_1 has split multiplicative reduction of type I_{n_1} and E_2 has good reduction. Then

$$\tau_p(A) = \tau_p(E_1) + \tau_p(E_2) = 1 + 0 = 1,$$

and

$$\Phi_{A,p} \simeq \Phi_{E_1,p} \times \Phi_{E_2,p} \simeq \mathbb{Z}/n_1\mathbb{Z}.$$

hence

$$L_p(A, s)^{-1} = (1 - p^{-s})^{\tau_p} P_p(p^{-s}), \quad \deg P_p = 2g - 2\tau_p = 2.$$

Thus the total degree is

$$\tau_p + \deg P_p = 1 + 2 = 3 = 2g - \tau_p,$$

in accordance with [Corollary 4.3](#). If both E_i are split multiplicative with types I_{n_i} , then $\tau_p(A) = 2$ and

$$\Phi_{A,p} \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}, \quad \deg P_p = 2g - 2\tau_p = 0.$$

The toric rank contributes the factor $(1 - p^{-s})^{\tau_p}$; after extracting it, the residual polynomial has degree $2g - 2\tau_p$, so the total invariant degree is $2g - \tau_p$ ([Corollary 4.3](#)), while $c_{A,p} = \#\Phi_{A,p}$ multiplies ([Proposition 2.8](#)).

Example 7.6 (Hecke control away from the level and isogeny degree). Let A be a T_ℓ -stable abelian quotient of $J_0(N)$, and suppose that the induced endomorphism

$$u_\ell: A \longrightarrow A$$

is an isogeny of degree d_ℓ . For every prime

$$p \nmid N\ell d_\ell,$$

the map u_ℓ extends to the Néron model of A , its kernel on the identity component is finite étale, and the induced map

$$\Phi_{A,p} \longrightarrow \Phi_{A,p}$$

has kernel and cokernel annihilated by d_ℓ .

For every auxiliary prime $r \neq p$, the isogeny induces an automorphism

$$V_r(A)^{I_p} \xrightarrow{\sim} V_r(A)^{I_p}$$

commuting with Frobenius. Thus the inertia-invariant local polynomial and the Artin conductor exponent of the fixed abelian variety A are compatible with the Hecke endomorphism. If the reduction is semistable, the induced isogeny on the maximal torus preserves its rank $\tau_{A,p}$.

If $p \mid N\ell d_\ell$, the finite-étale component-group argument need not apply. If u_ℓ is not an isogeny, no isogeny conclusion is drawn. In particular, the degeneracy maps are not themselves treated as isogenies between $J_0(N\ell)$ and $J_0(N)$, and no undefined “morphism factor” is used.

The preceding examples show where the tame hypotheses are used and where they fail. They also illustrate how the component-group bounds of Theorem 3.5 enter the local-factor comparisons.

8 Conclusion and Outlook

Synthesis. Working over the arithmetic base from Notation 2.1, the paper established a geometric-cohomological control principle for Néron models and traced its arithmetic fallout along three axes.

Chain I (existence/rigidity $\Rightarrow$ component groups $\Rightarrow$ local factors). By Theorem 3.1 the prime-to-residue-characteristic extension and uniqueness of isogenies in arithmetic families hold on a dense open. Upper semicontinuity of component groups in fibers (Theorem 3.5) then yields the local factor template (Corollary 4.3), which identifies the tame exponent τ_p and constrains $L_p(A, s)$. The mechanism appears concretely in Example 7.1. For modular Jacobians, Example 6.3 instead records the action of an explicitly defined Hecke endomorphism u_ℓ that is assumed to be an isogeny on the component group of a fixed T_ℓ -stable quotient.

Chain II (cohomology under base change $\Rightarrow$ Euler quotients $\Rightarrow$ conductor stability). The comparison Theorem 4.1 (items (i)–(iii)) proves invariance of I_p -invariants under prime-to- p isogenies on suitable opens, and quantifies possible jumps. This feeds directly into rational local L -factor invariance (Proposition 6.1) and conductor stability (Corollary 6.2), while the component-group estimates remain separate Tamagawa/local-defect controls. Example 6.4 shows the sharpness of the prime-to- p hypothesis.

Chain III (isogenies $\Rightarrow$ canonical-height functoriality; rank 0 $\Rightarrow$ conditional finiteness). Propositions 4.7 and 6.5 record the standard functoriality of canonical heights and the precisely defined tame geometric control of Definition 4.6. Separately, the rank-0 and finite-fiber hypotheses give the finiteness criterion of Corollary 6.6, illustrated by Example 6.7 and contrasted by Example 6.8.

Outlook

(A) Higher-dimensional generalizations. Two concrete directions arise naturally.

- *Semi-abelian extensions over parahoric level.* Replacing abelian by semi-abelian schemes on integral models with parahoric level structure (cf. Theorem 5.10) should keep the tame exponent τ_p stable on the same open loci and refine conductor bounds for boundary degenerations; see also the parahoric scope boundary in Example 5.13 for the additional hypotheses that would be required.
- *Families over Dedekind subschemes.* Passing from $S = \text{Spec } \mathbb{Z}$ to Dedekind subschemes of number fields does not alter the proofs in Theorems 3.1, 3.5 and 4.1; the local statements transport verbatim after replacing p by nonarchimedean places and tracking ramification indices.

(B) Derived and perverse sheaf refinements. The comparison in Theorem 4.1 can be lifted from invariants of $H_{\text{ét}}^1$ to a functorial statement for truncations of the vanishing-cycles complex $\mathcal{R}\Phi$ on the special fiber (cf. [2, Exposé XIII, pp. 82–115]). A derived-level control of the unipotent monodromy filtration would turn the inequalities in (iii) into equalities under test conditions (e.g. semistability plus a weight-monodromy constraint), providing sharper conductor formulas at wild primes.

(C) Motivic cohomology and regulators. The line $\omega_{A/S}$ in Construction 2.11 admits an interpretation via Deligne–Beilinson realizations. A comparison of the resulting regulators with the canonical heights used in Proposition 6.5 would clarify how far the finiteness mechanism in Corollary 6.6 extends beyond rank 0, for instance to situations with controlled positive rank but auxiliary height gaps on images of curves.

Mixed Tate and Hodge-theoretic scope. The Néron-model techniques used here control semistable realization data associated with abelian and semi-abelian degenerations, including the corresponding ℓ -adic inertia and differential data. They do not by themselves yield a general Hodge decomposition for arbitrary hypercohomology groups of Mixed Tate objects. Any extension to Mixed Tate motivic cohomology would therefore require compatibility of the relevant realizations with this semistable framework; no general Mixed-Tate realization theorem is asserted here.

(D) Hecke-stable quotient factors. A natural continuation is to determine, for each N and ℓ , the T_ℓ -stable quotient factors

$$A \text{ of } J_0(N)$$

on which the induced endomorphism u_ℓ is an isogeny, and to compute effectively:

- (i) the degree $\deg(u_\ell)$;
- (ii) the excluded primes dividing $N\ell \deg(u_\ell)$;

- (iii) the induced endomorphisms of the component groups $\Phi_{A,p}$; and
- (iv) the action of u_ℓ on $V_r(A)^{I_p}$.

Any equidistribution statement involving genuinely varying Hecke orbits would require a separately defined moduli family and a separate equidistribution theorem; it is not a consequence of the present Néron-model argument.

(E) Effective and computational aspects. The explicit cases in Examples 7.1 and 7.2 can be extended to large families, supplying numerical evidence for the bounds in Corollary 6.2 and for the persistence of finiteness in Corollary 6.6. A systematic implementation of the local-factor shape in Corollary 4.3, together with standard canonical-height computations, could support practical investigations of rational points on curves mapping to low-rank Jacobians.

Conjecture (Derived tame control). Under the semistability and weight–monodromy hypotheses described above, the tame kernel–cokernel control of Theorem 3.5 admits a functorial refinement on suitable graded pieces of the monodromy filtration of the vanishing-cycles complex $R\Phi$, compatible with prime-to-residue-characteristic isogenies.

(F) Open directions.

- **Derived vanishing-cycle refinements.** Conjecture 8 formulates the derived refinement suggested by Theorems 3.5 and 4.1. Establishing precise hypotheses under which the present uniform bounds become equalities remains open.
- **Extension of the fixed-quotient Hecke framework.** For which T_ℓ -stable quotients A of $J_0(N)$ is the induced endomorphism u_ℓ an isogeny, and how effectively can its degree and its action on $\Phi_{A,p}$ be determined?
- **Regulator and height comparisons.** For maps $\phi : C \rightarrow A$ as in Corollary 6.6, can regulator or height-gap hypotheses on $\phi(C)$ replace the rank-0 assumption while preserving the conditional finiteness conclusion?

Closing remark. The results above show that, on a common dense tame locus, prime-to-residue-characteristic isogenies give the stated control of component groups, inertia invariants, local Euler factors, and conductors. The height-theoretic conclusion consists of standard Néron–Tate functoriality together with the tame geometric control specified in Definition 4.6; no independent bound for individual local-height contributions is asserted.

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