

Homological Obstructions to Integral Closure in Simple Extensions of Valuation Rings

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Abstract

Let V be a valuation ring with fraction field K , let $L = K(\alpha)$ be a finite simple extension, and let $A = V[\alpha] \subseteq W$, where W denotes the integral closure of A in L . We study homological defect modules associated to the normalization quotient W/A using ordinary derived functors. For every valuation-compatible ideal $I \subseteq A$, we define derived integral closure defects

$$\mathrm{Tor}_i^A(W/A, A/I),$$

which measure the failure of reduction modulo I to preserve exactness of the normalization sequence

$$0 \rightarrow A \rightarrow W \rightarrow W/A \rightarrow 0.$$

The new content is the systematic valuation-theoretic use of these ordinary Tor groups as normalization-defect invariants: we identify their conductor support, isolate the connecting image as the actual obstruction to exact reduction, and compute their π -primary torsion profiles in DVR-order normalizations. In particular, in the DVR-order setting the first derived defect records the π -primary normalization-quotient thickness, including non-cyclic elementary-divisor profiles. In ramified DVR-order examples this profile may reflect ramified structure, but the Tor computation itself is an invariant of the normalization quotient rather than a ramification invariant. The approach is entirely algebraic and valuation-theoretic, using only classical homological methods and avoiding higher-categorical or spectral machinery.

Keywords: valuation rings; integral closure; simple extensions; derived functors; Tor modules; normalization defects; valuation-compatible reductions; ramification; conductor ideals.

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1 Introduction

Let V be a valuation ring with fraction field K , let $L = K(\alpha)$ be a simple finite extension, and put

$$A := V[\alpha].$$

Throughout the paper, unless explicitly stated otherwise, W denotes the integral closure of this order A in L , namely

$$W := \overline{A}^L.$$

Thus the normalization inclusion studied here is

$$A = V[\alpha] \subseteq W = \overline{A}^L.$$

This passage encodes subtle arithmetic and valuation-theoretic information: extension of valuations, ramification, conductor behaviour, and failure of finite generation in non-Noetherian settings.

Classically, the study of such extensions is governed by valuation prolongations, integral dependence, residual extensions, ramification indices, and Henselian lifting ([7, Ch. III, §§13,16–18, pp. 94–143; §§19–22, pp. 144–184]; [1, Ch. V, §1, nos. 1–2, pp. 303–308; Ch. VI, §8, nos. 1–2, pp. 416–418]). The

purpose of this article is to introduce a derived but still elementary homological diagnostic for this picture. Instead of using derived algebraic geometry, we use ordinary Tor and finite free resolutions to measure where valuation-compatible reductions of the normalization quotient fail to be exact.

Guiding principle. The word “derived” in this paper means homological and module-theoretic. It does not refer to ∞ -categories, spectral rings, or derived schemes.

Scope. The paper does not claim that the functor Tor or the flatness criterion for Tor-vanishing is new. Its novelty is instead the construction of a valuation-theoretic normalization diagnostic in which the ordinary Tor groups

$$\mathrm{Tor}_i^A(W/A, A/I)$$

are organized as defect modules attached to valuation-compatible reductions of the normalization sequence. The contribution has three concrete parts: first, the defects are localized on the conductor locus; second, the connecting image

$$\mathrm{Tor}_1^A(W/A, A/I) \longrightarrow A/I$$

is identified as the actual obstruction to preserving injectivity after reduction; and third, in DVR-order situations the first defect is computed in terms of the π -primary elementary-divisor profile of W/A . Thus the contribution is not an abstract new Tor formalism, but a computable valuation/conductor interpretation of Tor as a normalization-defect invariant.

Let

$$A := V[\alpha], \quad W := \overline{A}^L.$$

The inclusion $A \subseteq W$ gives the conductor

$$\mathfrak{c}_{A/W} := (A : W) = \{a \in A : aW \subseteq A\}.$$

The usual integral closure data records the quotient W/A . Our approach keeps track of the higher Tor invariants

$$\mathrm{Tor}_i^A(W/A, A/I)$$

for valuation-compatible ideals $I \subseteq A$.

1.1 Main contributions

The article develops the following diagnostic homological framework. The construction uses ordinary Tor groups, but the contribution is their valuation-theoretic interpretation and explicit computation in normalization problems.

- (i) We attach to $A = V[\alpha] \subseteq W$ the modules

$$\mathrm{Tor}_i^A(W/A, A/I),$$

not as a new homological functor, but as homological defect modules diagnosing exactness failure of normalization after valuation-compatible reduction.

- (ii) We identify their support and first obstruction behaviour along the conductor locus, especially through the connecting image

$$\mathrm{Tor}_1^A(W/A, A/I) \longrightarrow A/I.$$

- (iii) We compute the first defect in DVR-order situations, showing that it records the full π -primary torsion profile of W/A , including non-cyclic elementary-divisor examples.

These points are not merely formal consequences of the definition. The support result explains where the defects can live, the connecting-image criterion separates harmless Tor from genuine failure of

exact reduction, and the DVR-order formulas show that the first defect recovers more than the total length of W/A : it detects the full π -primary torsion profile visible through the functions

$$n \longmapsto \ell_V(\mathrm{Tor}_1^A(W/A, A/(\pi^n))).$$

This is the main reason the construction gives information beyond the single discriminant-index length: the resulting Tor-length profile can recover finer features of the A -module W/A , although it is itself functorially determined by W/A .

The modules

$$\mathrm{Tor}_i^A(W/A, A/I)$$

measure the failure of reduction modulo I to preserve exactness across the normalization quotient

$$0 \rightarrow A \rightarrow W \rightarrow W/A \rightarrow 0.$$

When the normalization quotient W/A is flat over A , the higher homological defect modules vanish. In ramified or conductor-singular situations, however, nontrivial Tor may appear and records homological deviation from exact integral closure descent; the actual obstruction to injectivity after reduction is the connecting image of $\mathrm{Tor}_1^A(W/A, A/I) \rightarrow A/I$.

2 Preliminaries on valuation rings and simple extensions

Throughout the paper, V denotes a valuation ring with fraction field K . Its maximal ideal is denoted by $\mathfrak{m}_V$, and its value group by Γ_V .

Definition 2.1 (Simple extension). Let L/K be a finite field extension. We say that L/K is simple over K if $L = K(\alpha)$ for some $\alpha \in L$. For such an element α , we put

$$A := V[\alpha] \subseteq L.$$

Definition 2.2 (Integral closure). The integral closure of A in L is

$$\overline{A}^L := \{x \in L : x \text{ is integral over } A\}.$$

When $A = V[\alpha]$, we write

$$W := \overline{V[\alpha]}^L.$$

([4, Def. 2.1.1, p. 23])

Remark 2.3. If A is Noetherian, then the integral closure W is finite over A whenever A is Japanese, while finite generation can fail in general. Classically, this finiteness phenomenon is already reflected in Nagata's theory of pseudo-geometric rings, where affine rings over pseudo-geometric integral domains have finite derived normal rings [5, (36.6), p. 133]; see also [3, §9, pp. 64–70]. In valuation-theoretic settings, especially non-Noetherian ones, this makes it useful to study normalization through filtrations and homological approximations rather than finite presentation alone.

Definition 2.4 (Conductor). For an extension of rings $A \subseteq W$, the conductor is

$$\mathfrak{c}_{A/W} := (A : W) = \{a \in A : aW \subseteq A\}.$$

([4, Def. 12.0.1, p. 238])

Definition 2.5 (Valuation-compatible ideal). Let $A = V[\alpha] \subseteq L$, and let v be a chosen prolongation of the valuation of V to L such that $A \subseteq \{x \in L : v(x) \geq 0\}$. For $\gamma \geq 0$, put

$$I_\gamma = \{a \in A : v(a) \geq \gamma\}.$$

An ideal $I \subseteq A$ is called valuation-compatible if it is generated by valuation-depth ideals I_γ , or is an ordinary power or finite product of such ideals.

Lemma 2.6 (Conductor quotient sequence). *Let $A \subseteq W$ be an extension of rings as above and let*

$$\mathfrak{c} = (A : W).$$

Then $\mathfrak{c}$ is an ideal of both A and W , and there is a natural exact sequence

$$0 \longrightarrow A/\mathfrak{c} \longrightarrow W/\mathfrak{c} \longrightarrow W/A \longrightarrow 0.$$

In particular, the quotient W/A is naturally controlled by the conductor layer $W/\mathfrak{c}$. Since $\mathfrak{c}$ is an ideal of W , one has

$$\mathfrak{c}W = \mathfrak{c}.$$

Consequently, the natural map

$$A/\mathfrak{c} \longrightarrow W/\mathfrak{c}W$$

is injective. Thus reduction modulo the conductor preserves injectivity, even though the groups

$$\mathrm{Tor}_i^A(W/A, A/\mathfrak{c})$$

may still be nonzero. ([4, Def. 12.0.1, p. 238])

Proof. Since

$$\mathfrak{c} = (A : W) = \{a \in A : aW \subseteq A\},$$

it is immediate that $\mathfrak{c}$ is an ideal of A . We now show that it is also an ideal of W . Let $x \in \mathfrak{c}$ and $w \in W$. Since $xW \subseteq A$, we have $wx \in A$. Moreover, for every $y \in W$, the product wy belongs to W , because W is a ring. Hence

$$(wx)y = x(wy) \in A.$$

Thus $(wx)W \subseteq A$, and therefore $wx \in \mathfrak{c}$. This proves that $\mathfrak{c}$ is an ideal of W .

The inclusion $A \hookrightarrow W$ induces an injective map

$$A/\mathfrak{c} \longrightarrow W/\mathfrak{c}.$$

Its cokernel is

$$(W/\mathfrak{c})/(A/\mathfrak{c}) \cong W/A,$$

which gives the claimed short exact sequence.

The sequence identifies W/A with the cokernel of the injective conductor-quotient map. Moreover, since $\mathfrak{c}W = \mathfrak{c}$, conductor reduction preserves injectivity. Consequently, a nonzero group $\mathrm{Tor}_1^A(W/A, A/\mathfrak{c})$ is homological information about the normalization quotient, but its connecting image in $A/\mathfrak{c}$ is zero. $\square$

Lemma 2.7 (Basic exact sequence). *For $A = V[\alpha] \subseteq W$, there is a canonical short exact sequence*

$$0 \longrightarrow A \longrightarrow W \longrightarrow W/A \longrightarrow 0.$$

The quotient module W/A measures the precise failure of A to be integrally closed in L . In particular, torsion and conductor support are reflected directly in the structure of W/A . In ramified order examples, these normalization defects may also reflect ramified structure, but W/A is not itself a ramification invariant.

Proof. The inclusion

$$A \hookrightarrow W$$

is the canonical embedding of A into its integral closure in L . Since the map is injective, its cokernel is the quotient module W/A , yielding the exact sequence.

The module W/A vanishes precisely when A is already integrally closed in L . More generally, its support detects where normalization fails, while its torsion structure records conductor-supported normalization defect. In ramified examples this torsion may be influenced by ramified structure, but the invariant being measured here is the normalization quotient W/A , not the ramification index itself. $\square$

Theorem 2.8 (Conductor support and localization criterion). *Let $A \subseteq B$ be an integral extension inside a finite field extension, and let $\mathfrak{c} = (A : B)$. If $\mathfrak{c} \neq 0$, then for every prime $\mathfrak{p} \in \text{Spec } A$ with $\mathfrak{p} \not\supseteq \mathfrak{c}$, one has*

$$A_{\mathfrak{p}} = B_{\mathfrak{p}}.$$

Consequently,

$$\text{Supp}_A(B/A) \subseteq V(\mathfrak{c}),$$

and for every ideal $I \subseteq A$ and every $i \geq 0$,

$$\text{Supp}_A \text{Tor}_i^A(B/A, A/I) \subseteq V(\mathfrak{c}).$$

If $\mathfrak{c} = 0$, the statement gives no informative support restriction: one only has the tautological containment

$$\text{Supp}_A(B/A) \subseteq \text{Spec } A.$$

Thus the conductor-support assertion is meaningful only on the nonzero-conductor locus.

Proof. Assume first that $\mathfrak{c} \neq 0$. Let $\mathfrak{p} \in \text{Spec } A$ with $\mathfrak{p} \not\supseteq \mathfrak{c}$. Choose $s \in \mathfrak{c} \setminus \mathfrak{p}$. Since $sB \subseteq A$, the element s becomes a unit in $A_{\mathfrak{p}}$. Hence every $b \in B_{\mathfrak{p}}$ satisfies

$$b = s^{-1}(sb) \in A_{\mathfrak{p}}.$$

Thus $B_{\mathfrak{p}} = A_{\mathfrak{p}}$. Therefore $(B/A)_{\mathfrak{p}} = 0$, proving the support inclusion. Localizing Tor gives

$$\text{Tor}_i^A(B/A, A/I)_{\mathfrak{p}} \cong \text{Tor}_i^{A_{\mathfrak{p}}}((B/A)_{\mathfrak{p}}, (A/I)_{\mathfrak{p}}) = 0.$$

If $\mathfrak{c} = 0$, there is no nonzero conductor element to invert, and the preceding argument gives no proper closed support bound. The remaining statement is only the tautological support containment in $\text{Spec } A$. $\square$

Lemma 2.9 (Support of the normalization defect). *Let $A = V[\alpha] \subseteq W$ be as above, and let*

$$\mathfrak{c} = (A : W)$$

be the conductor ideal. Then

$$\text{Supp}_A(W/A) \subseteq V(\mathfrak{c}).$$

In particular, the failure of integral closure is supported along the conductor locus.

Proof. This is the special case $i = 0$ of [Theorem 2.8](#). $\square$

3 Derived normalization-defect systems

We now introduce the homological object used throughout the paper. The definition uses ordinary Tor, but the point is the valuation-theoretic normalization problem to which it is applied: the quotient W/A , the valuation-compatible reductions A/I , the conductor support, and the explicit DVR-order torsion computations below together produce a computable defect system attached to the normalization $A \subseteq W$.

Definition 3.1 (Derived integral closure defect). Let $A \subseteq W$ be as above, and let $I \subseteq A$ be an ideal. The i -th derived integral closure defect of I is

$$\text{DIC}_i(A, W; I) := \text{Tor}_i^A(W/A, A/I).$$

Remark 3.2. The module $\text{DIC}_0(A, W; I)$ is the ordinary reduction

$$(W/A) \otimes_A A/I.$$

The higher modules $\text{DIC}_i(A, W; I)$, $i \geq 1$, measure failure of exact base change across the integral closure quotient.

Definition 3.3 (Derived integral-closure defect system). Let $I \subseteq A$. The derived integral-closure defect system associated to the powers of I is the family

$$\mathcal{D}_{\text{der}}^n(A, W; I) := \{\text{DIC}_i(A, W; I^n)\}_{i \geq 0, n \geq 0}.$$

The associated graded defect terms are

$$\text{gr}_{\text{der}}^n(A, W; I) := \{\text{Tor}_i^A(W/A, I^n/I^{n+1})\}_{i \geq 0}.$$

The inclusions $I^{n+1} \subseteq I^n$ induce transition morphisms between the corresponding quotient modules. Thus the above family is best viewed as a functorial inverse/direct system of Tor-modules, not as a filtration of one fixed module unless an additional filtration structure is specified.

Lemma 3.4 (Long exact Tor sequence). *For every ideal $I \subseteq A$, the short exact sequence*

$$0 \rightarrow A \rightarrow W \rightarrow W/A \rightarrow 0$$

induces an exact sequence

$$\text{Tor}_1^A(W, A/I) \rightarrow \text{Tor}_1^A(W/A, A/I) \rightarrow A/I \rightarrow W/IW \rightarrow (W/A) \otimes_A A/I \rightarrow 0.$$

Proof. Apply $-\otimes_A A/I$ to the short exact sequence and take the associated long exact Tor sequence. $\square$

Proposition 3.5 (Connecting image for an arbitrary reduction). *Let $I \subseteq A$, and let*

$$\delta_I : \text{Tor}_1^A(W/A, A/I) \rightarrow A/I$$

be the connecting homomorphism associated to

$$0 \rightarrow A \rightarrow W \rightarrow W/A \rightarrow 0.$$

Then

$$\text{im}(\delta_I) = \ker(A/I \rightarrow W/IW) \cong \frac{A \cap IW}{I},$$

where the intersection is taken inside W . Consequently, reduction modulo I preserves injectivity if and only if

$$A \cap IW = I.$$

In particular, for the conductor $\mathfrak{c} = (A : W)$, one has

$$\mathfrak{c}W = \mathfrak{c} \quad \text{and} \quad \text{im}(\delta_{\mathfrak{c}}) = 0.$$

Thus conductor reduction itself produces no injectivity obstruction.

Proof. The long exact Tor sequence contains

$$\text{Tor}_1^A(W/A, A/I) \xrightarrow{\delta_I} A/I \rightarrow W/IW.$$

Exactness gives

$$\text{im}(\delta_I) = \ker(A/I \rightarrow W/IW).$$

An element $a + I \in A/I$ belongs to this kernel precisely when $a \in IW$. Hence

$$\ker(A/I \rightarrow W/IW) = \frac{A \cap IW}{I}.$$

For $I = \mathfrak{c}$, the conductor is an ideal of W , so $\mathfrak{c}W = \mathfrak{c}$. Therefore

$$A \cap \mathfrak{c}W = \mathfrak{c},$$

and the stated connecting image is zero. $\square$

Proposition 3.6 (Support of derived defects). *Let*

$$\mathfrak{c} = (A : W).$$

Assume $\mathfrak{c} \neq 0$. Then for every ideal $I \subseteq A$ and every $i \geq 0$,

$$\mathrm{Supp}_A(\mathrm{DIC}_i(A, W; I)) \subseteq V(\mathfrak{c}).$$

In particular, on the nonzero-conductor locus, all derived integral closure defects are supported along the conductor locus.

Proof. Localize at a prime ideal $\mathfrak{p} \not\subseteq \mathfrak{c}$. By [Theorem 2.8](#),

$$W_{\mathfrak{p}} = A_{\mathfrak{p}}.$$

Hence

$$(W/A)_{\mathfrak{p}} = 0.$$

Therefore

$$\mathrm{Tor}_i^{A_{\mathfrak{p}}}((W/A)_{\mathfrak{p}}, (A/I)_{\mathfrak{p}}) = 0$$

for all $i \geq 0$. □

Remark 3.7 (Degree-zero term). For every ideal $I \subseteq A$,

$$\mathrm{DIC}_0(A, W; I) = (W/A) \otimes_A A/I,$$

because $\mathrm{Tor}_0^A(-, -) = - \otimes_A -$.

4 Exactness after reduction

Remark 4.1 (Flat case). If W/A is flat over A , then for every ideal $I \subseteq A$, every $n \geq 1$, and every $i \geq 1$,

$$\mathrm{DIC}_i(A, W; I^n) = \mathrm{Tor}_i^A(W/A, A/I^n) = 0.$$

Thus only the degree-zero reduction

$$\mathrm{DIC}_0(A, W; I^n) = (W/A) \otimes_A A/I^n$$

remains. This is the standard higher-Tor vanishing consequence of flatness.

Theorem 4.2 (First obstruction to exact reduction). *Let $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$ be the normalization sequence. For an ideal $I \subseteq A$, the kernel of the reduced map*

$$A/I \longrightarrow B/IB$$

is exactly the image of the connecting homomorphism

$$\delta : \mathrm{Tor}_1^A(B/A, A/I) \longrightarrow A/I.$$

Consequently, reduction modulo I preserves injectivity if and only if

$$\mathrm{im}(\delta) = 0.$$

In particular, a nonzero element of $\mathrm{Tor}_1^A(B/A, A/I)$ gives an actual failure of exact reduction only when its image under δ is nonzero.

Proof. Start with the canonical exact sequence associated to the normalization inclusion:

$$0 \longrightarrow A \longrightarrow W \longrightarrow W/A \longrightarrow 0.$$

Tensoring this sequence with A/I need not preserve left exactness, because tensor product is right exact but not generally exact. Applying the functor $- \otimes_A A/I$ and taking the associated long exact Tor sequence gives

$$\mathrm{Tor}_1^A(W, A/I) \longrightarrow \mathrm{Tor}_1^A(W/A, A/I) \xrightarrow{\delta} A \otimes_A A/I \longrightarrow W \otimes_A A/I \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0.$$

Using the natural identifications

$$A \otimes_A A/I \cong A/I, \quad W \otimes_A A/I \cong W/IW,$$

this becomes

$$\mathrm{Tor}_1^A(W, A/I) \longrightarrow \mathrm{Tor}_1^A(W/A, A/I) \xrightarrow{\delta} A/I \longrightarrow W/IW \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0.$$

The map

$$A/I \longrightarrow W/IW$$

is the reduction modulo I of the inclusion $A \hookrightarrow W$. Thus the reduced sequence

$$0 \longrightarrow A/I \longrightarrow W/IW \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0$$

is exact at A/I precisely when the connecting contribution from

$$\mathrm{Tor}_1^A(W/A, A/I)$$

does not create a nonzero kernel of $A/I \rightarrow W/IW$.

In particular, whenever

$$\mathrm{Tor}_1^A(W/A, A/I) \neq 0$$

and its image under the connecting homomorphism is nonzero, the map

$$A/I \longrightarrow W/IW$$

fails to be injective. Therefore reduction modulo I fails to preserve the left exactness of

$$0 \rightarrow A \rightarrow W \rightarrow W/A \rightarrow 0.$$

Thus the possible first obstruction is carried by the first derived integral-closure defect

$$\mathrm{DIC}_1(A, W; I) = \mathrm{Tor}_1^A(W/A, A/I),$$

but the actual obstruction to injectivity after reduction is its connecting image

$$\mathrm{im}(\delta : \mathrm{Tor}_1^A(W/A, A/I) \longrightarrow A/I).$$

Consequently, a nonzero class in $\mathrm{DIC}_1(A, W; I)$ produces a genuine failure of exact reduction only if its image under δ is nonzero. $\square$

5 Simple extensions and valuation prolongations

Throughout this section only, we also use the notation

$$\overline{V}^L := \{x \in L : x \text{ is integral over } V\}, \quad \overline{A}^L := \{x \in L : x \text{ is integral over } A\}.$$

The normalization used in the rest of the paper is always

$$W = \overline{A}^L,$$

where $A = V[\alpha]$. The valuative intersection over prolongations describes $\overline{V}^L$, not automatically $\overline{A}^L$. These two rings are identified only under an additional hypothesis, for example when $\overline{A}^L = \overline{V}^L$.

Let V be a valuation ring of K , and let $L = K(\alpha)$. A valuation ring W' of L lying over V , i.e. satisfying

$$W' \cap K = V,$$

is called a prolongation of V to L , in the classical sense of extensions of valuations to finite algebraic field extensions ([2, Ch. VI, §11, Thm. 19, pp. 50–58]; [6, Ch. 4, §4.1, pp. 107–118]; [7, Ch. III, §13, pp. 94–103]).

Definition 5.1 (Prolongation-compatible closure). Let V be a valuation ring of K , let L/K be a finite extension, and let

$$\mathcal{W}(V, L)$$

denote the set of valuation rings of L lying over V . Define the prolongation-compatible closure

$$W_{\text{val}} := \bigcap_{W' \in \mathcal{W}(V, L)} W'.$$

Equivalently, W_{val} consists precisely of those elements $x \in L$ which are contained in every prolongation of the valuation of V to L .

Remark 5.2. The ring W_{val} is intrinsically valuation-theoretic and is independent of the choice of generator α of the simple extension $L = K(\alpha)$. In particular, although the intermediate order

$$A = V[\alpha]$$

depends on the chosen primitive element, the valuative intersection W_{val} depends only on the extension of valued fields $(K, V) \subseteq (L, W')$.

The ring W_{val} therefore provides a canonical valuation-based description of the integral closure of V inside L .

Proposition 5.3 (Valuative description of integral closure). *Assume that L/K is finite. Then the integral closure of V in L satisfies*

$$\overline{V}^L = \bigcap_{W' \in \mathcal{W}(V, L)} W'.$$

In particular,

$$\overline{V}^L = W_{\text{val}}.$$

Proof. This is the classical valuative criterion for integral dependence over a valuation ring ([4, Thm. 6.8.3 and Prop. 6.8.14, pp. 136–141]; [3, §10, pp. 71–78]; [1, Ch. VI, §1, no. 3, p. 378]).

Indeed, every valuation ring W' lying over V is integrally closed. Hence every element integral over V belongs to each such W' , giving

$$\overline{V}^L \subseteq \bigcap_{W' \in \mathcal{W}(V, L)} W'.$$

Conversely, let $x \in L$ be not integral over V . By the valuative criterion for integrality applied relative to the valuation domain V , one may choose a valuation ring W' of L lying over V , equivalently

$$W' \cap K = V,$$

such that

$$x \notin W'.$$

Therefore

$$x \notin \bigcap_{W' \in \mathcal{W}(V, L)} W',$$

which proves the reverse inclusion. □

Remark 5.4 (Normalization versus prolongation). The preceding proposition describes the integral closure $\overline{V}^L$ of the valuation ring V in L . This should not be confused with the normalization

$$W = \overline{A}^L$$

of the chosen order $A = V[\alpha]$. In general, $\overline{V}^L$ and $\overline{A}^L$ need not coincide.

Only under the additional hypothesis

$$\overline{A}^L = \overline{V}^L$$

may the normalization quotient W/A be interpreted as the discrepancy between the chosen order A and the intersection of all prolongation-compatible valuation over-rings.

Construction 5.5 (Valuation-indexed ideals). Assume that the chosen prolongation v is compatible with A , in the sense that $A \subseteq \{x \in L : v(x) \geq 0\}$, and extend v by $v(0) = \infty$. For $\gamma \geq 0$, define

$$I_\gamma := \{a \in A : v(a) \geq \gamma\}.$$

Then I_γ is an ideal of A . The family

$$\{I_\gamma\}_{\gamma \in \Gamma_V}$$

defines a decreasing filtration of A indexed by the ordered value group Γ_V .

Moreover, if $\gamma' \geq \gamma$, then

$$I_{\gamma'} \subseteq I_\gamma.$$

Remark 5.6. The family $\{I_\gamma\}$ is the valuation-depth filtration associated with the chosen prolongation v ; in general, it need not coincide with an adic filtration of A .

If V is a DVR with uniformizer π , then, for every $n \geq 0$,

$$I_{nv(\pi)} = A \cap \pi^n \mathcal{O}_v, \quad \pi^n A \subseteq I_{nv(\pi)}.$$

At these indices, the valuation-depth filtration coincides with the π -adic filtration precisely under the explicit saturation condition

$$A \cap \pi^n \mathcal{O}_v = \pi^n A \quad \text{for every } n \geq 0.$$

Without this saturation condition, the ideals I_γ should be regarded as valuation-depth ideals rather than as the terms of the π -adic filtration. In higher-rank or non-discrete valuation settings, the full valuation-depth filtration also records information coming from the ordered value group.

Definition 5.7 (Valuation-indexed derived defect). For $\gamma \in \Gamma_V$, define the i -th valuation-indexed derived defect by

$$\text{Def}_i(\gamma) := \text{Tor}_i^A(W/A, A/I_\gamma).$$

Remark 5.8. The module

$$\text{Def}_0(\gamma) = (W/A) \otimes_A A/I_\gamma$$

records the ordinary valuation-theoretic reduction of the normalization defect modulo valuation depth γ .

The higher modules

$$\text{Def}_i(\gamma), \quad i \geq 1,$$

measure the failure of this reduction process to behave exactly from the homological point of view.

Thus the higher valuation-indexed defects record additional higher Tor data associated with the normalization quotient, which are invisible in the classical valuation filtration alone.

Theorem 5.9 (Valuation stratification of derived defects). *The assignment*

$$\gamma \longmapsto \text{Def}_i(\gamma)$$

is functorial with respect to inclusions

$$I_{\gamma'} \subseteq I_\gamma \quad (\gamma' \geq \gamma).$$

Hence the derived integral closure defects form a functorial system of Tor-modules indexed by the ordered value group Γ_V .

Moreover, the transition morphisms

$$\mathrm{Def}_i(\gamma') \longrightarrow \mathrm{Def}_i(\gamma)$$

describe how higher normalization Tor data vary as one passes to deeper valuation layers.

Proof. If $\gamma' \geq \gamma$, then by construction

$$I_{\gamma'} \subseteq I_{\gamma}.$$

Hence there is a natural quotient morphism

$$A/I_{\gamma'} \longrightarrow A/I_{\gamma}.$$

Since the functor

$$\mathrm{Tor}_i^A(W/A, -)$$

is covariant in its second argument, applying it yields natural transition maps

$$\mathrm{Tor}_i^A(W/A, A/I_{\gamma'}) \longrightarrow \mathrm{Tor}_i^A(W/A, A/I_{\gamma}).$$

These are precisely the morphisms

$$\mathrm{Def}_i(\gamma') \longrightarrow \mathrm{Def}_i(\gamma).$$

Thus the collection

$$\{\mathrm{Def}_i(\gamma)\}_{\gamma \in \Gamma_V}$$

forms a directed functorial system of Tor-modules indexed by the ordered value group. $\square$

Remark 5.10. The preceding viewpoint is purely algebraic. The ordered parameter space is the valuation group Γ_V , and the transition maps arise only from valuation-ideal inclusions and functoriality of ordinary Tor. No topological persistence module, spectral sequence, or obstruction tower is being constructed here.

5.1 π -primary normalization thickness and derived defects

Theorem 5.11 (General π -primary formula for finite-length normalization defects). *Let V be a DVR with uniformizer π , and let $A = V[\alpha] \subseteq W$ be as above. Assume that π is a non-zero-divisor on A , and put $M := W/A$. Whenever elementary-divisor decompositions or V -lengths are invoked below, we also assume that M is a finite-length π -primary V -module; equivalently, in the DVR-order applications under consideration, W/A is a finite V -module killed by a power of π . For every $n \geq 1$, there is a natural identification*

$$\mathrm{DIC}_1(A, W; (\pi^n)) = \mathrm{Tor}_1^A(M, A/(\pi^n)) \cong M[\pi^n],$$

where

$$M[\pi^n] := \{m \in M : \pi^n m = 0\}.$$

Here and throughout this subsection, all elementary-divisor decompositions and all lengths are taken over the DVR V , not over A .

In particular, if M is a finite-length π -primary V -module and

$$M \cong \bigoplus_{j=1}^s V/(\pi^{r_j})$$

as a V -module, then

$$\ell_V(\mathrm{DIC}_1(A, W; (\pi^n))) = \sum_{j=1}^s \min(r_j, n).$$

Thus the first derived normalization defect detects the full π -primary torsion profile of W/A , not only the cyclic case.

Proof. Since π is a non-zero-divisor on A , the quotient $A/(\pi^n)$ has the two-term free resolution

$$0 \longrightarrow A \xrightarrow{\pi^n} A \longrightarrow A/(\pi^n) \longrightarrow 0.$$

Tensoring this resolution with $M = W/A$ gives

$$0 \longrightarrow \mathrm{Tor}_1^A(M, A/(\pi^n)) \longrightarrow M \xrightarrow{\pi^n} M \longrightarrow M \otimes_A A/(\pi^n) \longrightarrow 0.$$

Hence

$$\mathrm{Tor}_1^A(M, A/(\pi^n)) \cong \ker(M \xrightarrow{\pi^n} M) = M[\pi^n].$$

If $M \cong \bigoplus_{j=1}^s V/(\pi^{r_j})$, then

$$(V/(\pi^{r_j}))[\pi^n] \cong V/(\pi^{\min(r_j, n)}).$$

Adding lengths over all summands gives

$$\ell_V(\mathrm{DIC}_1(A, W; (\pi^n))) = \sum_{j=1}^s \min(r_j, n).$$

This proves the formula. □

Theorem 5.12 (Cyclic specialization of the π -primary normalization-thickness formula). *Let V be a DVR with uniformizer π , and let $A = V[\alpha] \subseteq W$ be a DVR-order normalization extension. Assume that π is a non-zero-divisor on A . Suppose, in the cyclic finite-length π -primary case, that*

$$W/A \cong V/(\pi^r)$$

as a V -module.

Then, for every $n \geq 1$, the first derived defect

$$\mathrm{DIC}_1(A, W; (\pi^n)) = \mathrm{Tor}_1^A(W/A, A/(\pi^n))$$

is given by the explicit π -primary normalization-thickness formula

$$\mathrm{DIC}_1(A, W; (\pi^n)) \cong V/(\pi^{\min(r, n)})$$

as a V -module. Equivalently,

$$\ell_V(\mathrm{DIC}_1(A, W; (\pi^n))) = \min(r, n).$$

Thus the computation detects the π -primary torsion thickness of the normalization quotient W/A . In ramified DVR-order examples this thickness may reflect ramified structure, but the formula is a Tor computation for W/A , not a formula for the ramification index.

Proof. This theorem is the cyclic one-summand specialization of [Theorem 5.11](#). Put $M := W/A$. By [Theorem 5.11](#), for every $n \geq 1$,

$$\mathrm{DIC}_1(A, W; (\pi^n)) = \mathrm{Tor}_1^A(M, A/(\pi^n)) \cong M[\pi^n]$$

as a V -module, where

$$M[\pi^n] = \{m \in M : \pi^n m = 0\}.$$

Under the cyclic V -module hypothesis

$$M = W/A \cong V/(\pi^r),$$

one has

$$M[\pi^n] \cong (V/(\pi^r))[\pi^n] \cong V/(\pi^{\min(r, n)}).$$

Therefore

$$\mathrm{DIC}_1(A, W; (\pi^n)) \cong V/(\pi^{\min(r, n)})$$

as a V -module, and hence

$$\ell_V(\mathrm{DIC}_1(A, W; (\pi^n))) = \min(r, n).$$

This proves that the first derived defect detects the cyclic π -primary normalization thickness of W/A . The computation is a Tor computation for the normalization quotient W/A , not a formula for the ramification index. □

Corollary 5.13 (Vanishing converse for cyclic π -primary defect layers). *Let V be a DVR with uniformizer π , and let $A = V[\alpha] \subseteq W$ be a DVR-order normalization extension. Assume that π is a non-zero-divisor on A and that*

$$W/A \cong A/(\pi^r)$$

as an A -module for some $r \geq 0$. Then

$$\mathrm{DIC}_1(A, W; (\pi)) = 0 \iff r = 0 \iff W = A.$$

Equivalently, in this cyclic A -module situation, the first derived defect vanishes modulo (π) if and only if the normalization defect has zero π -primary thickness.

Proof. Since π is a non-zero-divisor on A , the two-term free resolution

$$0 \longrightarrow A \xrightarrow{\pi^r} A \longrightarrow A/(\pi^r) \longrightarrow 0$$

gives, for every $n \geq 1$,

$$\mathrm{DIC}_1(A, W; (\pi^n)) = \mathrm{Tor}_1^A(A/(\pi^r), A/(\pi^n)) \cong A/(\pi^{\min(r, n)}).$$

Taking $n = 1$, if $r \geq 1$, then

$$\mathrm{DIC}_1(A, W; (\pi)) \cong A/(\pi) \neq 0.$$

Hence $\mathrm{DIC}_1(A, W; (\pi)) = 0$ forces $r = 0$. Conversely, if $r = 0$, then $W/A \cong A/(\pi^0) = A/(1) = 0$, so $W = A$, and therefore $\mathrm{DIC}_1(A, W; (\pi)) = 0$. $\square$

Theorem 5.14 (DVR-order discriminant-index length formula for the normalization defect). *Let V be a DVR with uniformizer π , fraction field K , and residue field k . Let L/K be a finite separable extension, and let W be the maximal V -order in L . Let*

$$A = V[\alpha] \subseteq W$$

be a finite V -order.

This is a strictly DVR-order statement. It is not asserted in the general valuation-ring setting of Sections 2–6, where the relevant normalizations may be non-finite, non-free, or outside the finite Noetherian order framework required for the discriminant-index formula.

Let

$$\mathfrak{P}_1, \dots, \mathfrak{P}_g$$

be the maximal ideals of W lying over the maximal ideal of V . For each j , write

$$e_j = e(\mathfrak{P}_j/V), \quad f_j = [W/\mathfrak{P}_j : k].$$

Assume throughout this theorem that A and W are finite free V -modules of rank $[L : K]$, that W/A has finite V -length, and that both discriminants $\Delta_{A/V}$ and $\Delta_{W/V}$ are computed with respect to the trace pairing

$$(x, y) \mapsto \mathrm{Tr}_{L/K}(xy).$$

Write

$$d := v(\Delta_{A/V}).$$

Assume that the maximal order discriminant is tame at every prime $\mathfrak{P}_j$ above V , so that

$$v(\Delta_{W/V}) = \sum_{j=1}^g f_j(e_j - 1).$$

Then the V -length of the normalization defect is

$$\ell_V(W/A) = \frac{d - \sum_{j=1}^g f_j(e_j - 1)}{2}.$$

In the special case where W is a DVR, equivalently where there is a single prolongation of V to L , this reduces to

$$\ell_V(W/A) = \frac{d - f(e - 1)}{2}.$$

Consequently, the discriminant determines the total normalization-defect length, not the full π -primary torsion profile of W/A .

If, in addition, the normalization quotient is cyclic as a V -module,

$$W/A \cong V/(\pi^r)$$

for some $r \geq 0$, then

$$r = \ell_V(W/A) = \frac{d - \sum_{j=1}^g f_j(e_j - 1)}{2},$$

and [Theorem 5.12](#) gives

$$\ell_V(\text{DIC}_1(A, W; (\pi^n))) = \min \{n, \ell_V(W/A)\}.$$

Proof. Under the DVR-order finiteness hypotheses imposed in the statement, namely that A and W are finite free V -modules of the same rank and that W/A has finite V -length, the classical discriminant-index relation gives

$$v(\Delta_{A/V}) = v(\Delta_{W/V}) + 2 \ell_V(W/A).$$

Since the maximal order is assumed tame at every prime $\mathfrak{P}_j$ lying over V , the tame discriminant formula gives

$$v(\Delta_{W/V}) = \sum_{j=1}^g f_j(e_j - 1).$$

Thus

$$d = \sum_{j=1}^g f_j(e_j - 1) + 2 \ell_V(W/A),$$

and therefore

$$\ell_V(W/A) = \frac{d - \sum_{j=1}^g f_j(e_j - 1)}{2}.$$

This identity determines only the total V -length of the normalization quotient W/A . It does not by itself determine the full elementary-divisor or π -primary torsion profile of W/A , and hence it does not, in general, determine every Tor-defect length $\ell_V(\text{DIC}_1(A, W; (\pi^n)))$.

Under the additional cyclic V -module condition

$$W/A \cong V/(\pi^r),$$

one has

$$r = \ell_V(W/A).$$

The stated formula for $\ell_V(\text{DIC}_1(A, W; (\pi^n)))$ then follows directly from [Theorem 5.12](#). □

Corollary 5.15 (Different-index length formula for the normalization defect). *Let V be a Henselian DVR with uniformizer π , fraction field K , and residue field k . Let L/K be a finite separable extension such that the integral closure W of V in L is a DVR. Let*

$$\mathfrak{D}_{W/V}$$

denote the different ideal, and write

$$\mathfrak{D}_{W/V} = \mathfrak{p}_W^\delta.$$

Let $f = [k_W : k]$, and let $A = V[\alpha] \subseteq W$ be a finite V -order. Put

$$d_A := v(\Delta_{A/V}).$$

Then

$$\ell_V(W/A) = \frac{d_A - f\delta}{2}.$$

If, in addition,

$$W/A \cong V/(\pi^r)$$

as a V -module for some $r \geq 0$, then

$$r = \frac{d_A - f\delta}{2},$$

and for every $n \geq 1$,

$$\ell_V(\text{DIC}_1(A, W; (\pi^n))) = \min \{n, \ell_V(W/A)\}.$$

Proof. For a finite separable extension of DVRs, the discriminant of the maximal order is the norm of the different:

$$\Delta_{W/V} = N_{W/V}(\mathfrak{D}_{W/V}).$$

Since

$$\mathfrak{D}_{W/V} = \mathfrak{p}_W^\delta,$$

we obtain

$$v(\Delta_{W/V}) = f\delta.$$

The discriminant-index relation for the finite order $A \subseteq W$ gives

$$v(\Delta_{A/V}) = v(\Delta_{W/V}) + 2\ell_V(W/A).$$

Therefore

$$d_A = f\delta + 2\ell_V(W/A),$$

and hence

$$\ell_V(W/A) = \frac{d_A - f\delta}{2}.$$

The final Tor formula is not a consequence of the different alone. It follows only after imposing the additional cyclic V -module condition $W/A \cong V/(\pi^r)$, in which case it is exactly the cyclic specialization of [Theorem 5.12](#). $\square$

6 Scope in Henselian defectless settings

Definition 6.1 (Defectless extension). A finite extension L/K of valued fields is called defectless if the usual equality

$$[L : K] = \sum_j e_j f_j$$

holds over the extensions of the valuation from K to L ([6, Ch. 6, §6.1–§6.2, pp. 151–171]; [7, Ch. III, §18, pp. 136–143]).

Remark 6.2 (Conditional role of defectless Henselian hypotheses). Let V be a Henselian valuation ring and let $L = K(\alpha)$ be a finite defectless simple extension. Let $A = V[\alpha]$, and let W be the integral closure of A in L . The defectless and Henselian hypotheses are used here only as valuation-theoretic background conditions controlling the classical extension of valuations. They do not, by themselves, imply that the normalization quotient W/A is flat, nor do they imply

$$\text{Tor}_i^A(W/A, A/I_\gamma) = 0 \quad (i \geq 1).$$

Consequently, any vanishing statement for the valuation-indexed derived defects

$$\text{Def}_i(\gamma) = \text{Tor}_i^A(W/A, A/I_\gamma)$$

must be read as conditional on the explicit Tor-independence hypothesis $\text{Tor}_i^A(W/A, A/I_\gamma) = 0$ for all $i \geq 1$; a sufficient stronger condition is that W/A be flat over A .

7 Examples and computations

Example 7.1 (Main computed example: non-normal DVR order). Let k be a field with $\text{char } k \neq 2$, let $V = k[[\pi]]$, and put $K = k((\pi))$. Let $L = K(u)$ with $u^2 = \pi$. Then $W = k[[u]]$ is the integral closure of V in L .

Set

$$A = V[u^3] = k[[u^2, u^3]] \subsetneq W.$$

Since $u = u^3/u^2$, one has $L = K(u^3)$, so this is still a simple extension order. The quotient is concrete:

$$W/A \cong k \cdot \bar{u}.$$

Its annihilator in A is the maximal ideal

$$\mathfrak{m} = (u^2, u^3),$$

so

$$W/A \cong A/\mathfrak{m}.$$

Indeed, every element of $W = k[[u]]$ has a unique expansion

$$a_0 + a_1 u + a_2 u^2 + \cdots.$$

Since $A = k[[u^2, u^3]]$ contains 1 and all powers u^n for $n \geq 2$, the only missing monomial modulo A is u . Hence

$$W/A \cong k \cdot u$$

as a k -vector space. Moreover $u^2 u = u^3 \in A$ and $u^3 u = u^4 \in A$, so $\mathfrak{m} = (u^2, u^3)$ annihilates the class of u . Conversely, if $a \in A$ has nonzero constant term, then $au \notin A$. Hence the annihilator is exactly $\mathfrak{m}$, and therefore $W/A \cong A/\mathfrak{m}$.

Therefore

$$\text{DIC}_1(A, W; \mathfrak{m}) = \text{Tor}_1^A(A/\mathfrak{m}, A/\mathfrak{m}) \cong \mathfrak{m}/\mathfrak{m}^2,$$

which is nonzero. In fact, $\mathfrak{m}/\mathfrak{m}^2$ is generated by the classes of u^2 and u^3 .

Here

$$\mathfrak{m} = (A : W) \quad \text{and} \quad \mathfrak{m}W = \mathfrak{m}.$$

Consequently,

$$A/\mathfrak{m} \longrightarrow W/\mathfrak{m}W$$

is injective, and the connecting image of the displayed nonzero Tor module is zero. Thus this example gives nonzero conductor-supported homological information, not a conductor-reduction obstruction.

Example 7.2 (Template computation: two elementary-divisor normalization defect). Let V be a DVR with uniformizer π , and let $A = V[\alpha] \subseteq W$ be a finite DVR-order normalization extension such that π is a non-zero-divisor on A . Assume that the normalization quotient has two elementary divisors:

$$W/A \cong V/(\pi^2) \oplus V/(\pi^5)$$

as a finite-length π -primary V -module.

For $I = (\pi^n)$, the first derived normalization defect is

$$\text{DIC}_1(A, W; (\pi^n)) = \text{Tor}_1^A(W/A, A/(\pi^n)).$$

By the general π -primary torsion-profile formula of [Theorem 5.11](#), we obtain

$$\ell_V(\text{DIC}_1(A, W; (\pi^n))) = \min(2, n) + \min(5, n).$$

Equivalently, the values are

$$\ell_V(\text{DIC}_1(A, W; (\pi^n))) = \begin{cases} 2n, & 1 \leq n \leq 2, \\ n + 2, & 3 \leq n \leq 5, \\ 7, & n \geq 5. \end{cases}$$

Thus the first derived defect detects two distinct normalization thickness layers, not merely the total length $\ell_V(W/A) = 7$. In particular, this example shows explicitly why the derived defect profile contains more information than the single discriminant-index length alone.

Example 7.3 (Orientation: ramified cubic extension with conditional defect). Let $V = \mathbb{Z}_{(3)}$, $K = \mathbb{Q}$, and $L = \mathbb{Q}(\theta)$, where $\theta^3 = 3$. The polynomial $x^3 - 3$ is Eisenstein at 3, so the extension is totally ramified at 3. Put $A = V[\theta]$, and let B be the integral closure of A in L .

If $A = B$, then all derived defects vanish. If instead one replaces A by a non-normal V -order $A' \subsetneq B$, then

$$\mathrm{DIC}_1(A', B; (3)) = \mathrm{Tor}_1^{A'}(B/A', A'/(3))$$

records homological information associated with reduction at the ramified prime. An actual failure of injectivity occurs only when the image of this Tor module under the connecting homomorphism is nonzero.

[Example 7.1](#) gives a concrete non-normal DVR-order computation, while [Example 7.2](#) displays the two-layer Tor-length profile arising from the explicitly assumed elementary divisors 2 and 5 of W/A .

8 Concluding remarks

The construction introduced here gives a homological refinement of integral closure in simple extensions of valuation rings. Its strength is that it remains close to classical valuation theory while detecting additional failure of exactness through ordinary derived functors. This makes the framework suitable for valuation-theoretic readers while still allowing derived techniques to enter in a controlled and computational form.

Future work may include:

- (i) extension of the discriminant-index comparison from the π -primary torsion-profile formula to non- π -primary and mixed primary normalization defects;
- (ii) comparison, in examples where additional ramification data are present, of the resulting normalization-defect torsion profiles with higher ramification filtrations, Newton polygons, and higher different ideals;
- (iii) extension to towers of simple extensions;
- (iv) interaction with Henselian lifting and defect extensions.

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