

The UOF Hypothesis: Uncomputable Order Fractality and the Algorithmic Dimension of Permuted Partial Sums

Vladislav Vitalyevich Tishkov

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Abstract

We formulate a new hypothesis at the intersection of fractal geometry, the theory of divergent series, and algorithmic information theory. The hypothesis asserts the existence of a computable sequence whose partial-sum graph is fractal, whose Hausdorff dimension changes under permutation of its terms, and for which the resulting dimension function is algorithmically non-computable. Unlike previous formulations, we provide a constructive ansatz for such a sequence based on rapidly growing phases encoding the halting set. We give a fully detailed mathematical formulation, discuss its components, and connect the hypothesis to existing results and to several recent works of the author. The present work also includes an extensive discussion of possible proof strategies and the main technical obstacles that must be overcome. Two complementary approaches are developed: a topological-existential strategy yielding a weakened form, and an analytic-constructive strategy that, conditional on a new dimensional growth lemma, would establish the hypothesis in full generality.

1 Introduction

The classical Riemann rearrangement theorem shows that the order of summation of a conditionally convergent series can change its limit or even produce divergence. Fractal geometry provides natural examples of graphs of partial sums with non-integer Hausdorff dimension. Algorithmic information theory gives rigorous meaning to the notion of non-computability of numerical invariants. The present work proposes a single hypothesis, referred to as the **UOF hypothesis**, which combines these three phenomena.

The central philosophical motivation behind the UOF hypothesis is the conviction that order is not merely a syntactic property of a sequence, but a geometric one. In classical analysis, the order of summation is treated as a tool for convergence control. However, if we visualise the partial sums as a path in the complex plane, different orders produce different paths — and these paths can have radically different geometric complexities. The Riemann rearrangement theorem tells us that the limit can change; the UOF hypothesis goes further and claims that the intrinsic fractal dimension of the path can change, and moreover, that this change can encode algorithmic unsolvability. This transforms the classical rearrangement phenomenon from a curiosity of convergence into a bridge between geometry and computability theory.

The core novelty of the present formulation is that we do not merely claim existence — we propose a specific constructive mechanism: the phase of each term is chosen to encode the bits of a non-computable set (e.g., the halting set), while the modulus grows sufficiently fast to generate fractal geometry. This constructive nature distinguishes the UOF hypothesis from purely existential statements and makes it amenable to detailed mathematical analysis.

2 Main Conjecture

Conjecture 2.1 (UOF Hypothesis). *There exists a computable sequence $(a_k)_{k \in \mathbb{N}}$ of complex numbers such that:*

(roman*) the closure of the partial-sum graph of the natural order has Hausdorff dimension strictly between 1 and 2;

(roman*) there exist computable permutations π_1, π_2 of $\mathbb{N}$ such that the Hausdorff dimensions of the corresponding permuted partial-sum graphs are different;

(roman*) the function $\pi \mapsto D(\pi)$ is not algorithmically computable;

(roman*) moreover, there exists a threshold d_0 such that the set

$$\{\pi \in \Pi_{\text{comp}} : D(\pi) > d_0\}$$

is algorithmically undecidable.

Here Π_{comp} denotes the set of computable permutations of $\mathbb{N}$.

It is important to clarify what is meant by the partial-sum graph. For a sequence (a_k) and a permutation π , we define the partial sums

$$S_n^{(\pi)} = \sum_{k=1}^n a_{\pi(k)}.$$

The partial-sum graph is the set

$$\Gamma_\pi = \{(n, S_n^{(\pi)}) : n \in \mathbb{N}\} \subset \mathbb{R}^3,$$

or its projection onto the complex plane if we consider only the values $S_n^{(\pi)}$. The Hausdorff dimension is taken with respect to the Euclidean metric on $\mathbb{R}^3$ or $\mathbb{C}$. The closure is necessary to obtain a compact set, but in practice the asymptotic behaviour of the sequence determines the dimension.

3 Constructive Ansatz for the Sequence

To make the hypothesis amenable to proof, we propose the following explicit construction. Let $\Omega \subset \mathbb{N}$ be a non-computable set (e.g., the halting set). Define:

$$a_k = \begin{cases} k^2 \cdot e^{2\pi i \theta_k}, & k \in \Omega, \\ k^{-2} \cdot e^{2\pi i \phi_k}, & k \notin \Omega, \end{cases}$$

where θ_k and ϕ_k are computable sequences of angles chosen so that:

- the partial sums for $k \in \Omega$ form a self-similar Cantor-like set in the complex plane;
- the partial sums for $k \notin \Omega$ form a rectifiable curve;
- the ratio of growth rates ensures that the Hausdorff dimension of the total graph is controlled by the subset Ω .

The choice of k^2 for $k \in \Omega$ and k^{-2} for $k \notin \Omega$ is not arbitrary. It reflects a deliberate separation of scales: the large terms create jumps in the partial-sum path, while the small terms produce dense, oscillatory filling between those jumps. If the large terms are placed according to a non-computable set, then the positions of the jumps themselves encode algorithmic information. The phases θ_k and ϕ_k are then chosen to ensure that the jumps do not cancel each other out — for instance, by making them point in mutually orthogonal or quasi-random directions. This guarantees that the resulting set is not a simple curve but a genuine fractal with a dimension that depends on the density of large jumps. The computability of the entire sequence is preserved because the membership in Ω can be approximated, but the exact dimension requires exact knowledge of Ω , which is unattainable algorithmically.

This ansatz has two key features:

1. The sequence (a_k) is computable if Ω is replaced by its computable approximation, but the exact geometric dimension depends on the true membership in Ω .
2. Any permutation that moves elements of Ω changes the relative density of the two growth scales, thereby altering the Hausdorff dimension.

4 Components of the Hypothesis

4.1 Fractal Base Series

The first condition requires:

$$1 < D(\text{id}) < 2.$$

This means that the natural order already produces a genuinely fractal graph rather than a rectifiable curve or a space-filling set. In our ansatz, this holds if the set Ω has positive density and the angles θ_k are chosen to avoid accidental cancellations.

A crucial subtlety is that the graph of partial sums is not a standard fractal set like the Cantor set or the Koch curve: it is a sequence of points in $\mathbb{R}^3$ (if we count the index as a coordinate) or in $\mathbb{C}$ (if we only consider the complex plane). The Hausdorff dimension of such a graph is determined by the scaling of the maximal jumps and the density of their distribution. In our construction, the large terms k^2 act as spikes that dominate the local geometry. If the indices of these spikes form a set with positive density, the graph cannot be rectifiable. However, if the phases are poorly chosen, the spikes might cluster in a way that creates a space-filling curve, pushing the dimension to 2. The challenge is to tune the phases so that the dimension lies strictly between 1 and 2 — for example, by making the spikes follow a logarithmic spiral or a Cantor-like angular distribution.

4.2 Permutation Dependence

The second condition asserts that the Hausdorff dimension is not invariant under rearrangement. This is a geometric analogue of the Riemann rearrangement phenomenon. In our construction, a permutation that moves a positive proportion of Ω -elements to later positions changes the asymptotic scaling of the partial sums.

It is natural to ask: why should a permutation change the fractal dimension at all? After all, the set of terms is the same; only their order differs. The answer lies in the fact that the partial-sum graph is an ordered object: its geometry depends on the sequence of cumulative sums. If we move a large term (say, k^2) from an early position to a late one, the jump it creates occurs later, which alters the local scaling of the path. If we move many such terms, the global scaling exponent — and hence the Hausdorff dimension — can change. This is analogous to how rearranging a conditionally convergent series changes its sum, but here the invariant is not a numerical value but a geometric dimension. The effect is particularly dramatic when the permuted set has a different asymptotic density of large terms, because the dimension is a function of that density.

4.3 Algorithmic Non-computability

The third condition states that there is no algorithm which, given a computable description of a permutation π , outputs $D(\pi)$ to arbitrary precision. This follows from our ansatz because computing $D(\pi)$ would require deciding, for infinitely many k , whether $k \in \Omega$, which is equivalent to solving the halting problem.

The connection to the halting problem is more subtle than it might first appear. The dimension $D(\pi)$ is not simply a function of the characteristic function of Ω — it depends on the order in which Ω -elements appear under π . However, if π is computable, then the positions

of Ω -elements under π are still determined by Ω . To compute $D(\pi)$ exactly, one would need to know, for arbitrarily large k , whether $k \in \Omega$. This is precisely the halting problem. But there is a deeper issue: even if one could approximate $D(\pi)$ from below, the convergence rate would encode information about the density of Ω , which is itself non-computable. Thus, the non-computability is not merely an artifact of the definition — it is intrinsic to the geometric construction. This makes the UOF hypothesis a rare example of a geometric invariant that is algorithmically inaccessible.

The fourth condition is stronger: it says that the decision problem associated with a dimensional threshold is undecidable. This implies that the geometric complexity of the permuted sums carries algorithmic information. The threshold condition is particularly striking because it moves the undecidability from a function to a set. It says that even the qualitative question “does this permutation produce a dimension greater than d_0 ?” is undecidable. This is reminiscent of Rice’s theorem in computability theory, which states that any non-trivial property of computable functions is undecidable. Here, the property $D(\pi) > d_0$ is a non-trivial geometric property of permutations, and our construction suggests that it is undecidable because it depends on the halting set. This elevates the UOF hypothesis from a curiosity to a potential cornerstone of a new field: algorithmic fractal geometry.

5 Strategic Approaches to Proving the UOF Hypothesis

While the UOF hypothesis is formulated as a conjecture, we present here two distinct and complementary strategies for its resolution. The first approach is topological and existential, providing a relatively straightforward path to a weakened form of the hypothesis. The second approach is analytic and constructive, aiming for a full proof but requiring a deep new lemma connecting Hausdorff dimension to the growth of partial sums.

5.1 Strategy A: The Topological–Existential Approach

This approach seeks to prove the existence of a sequence satisfying the UOF hypothesis without explicitly computing the Hausdorff dimension for all permutations. It relies on two key observations: first, that the Hausdorff dimension is stable under finite perturbations of a sequence; second, that the halting set can be encoded into the asymptotic density of large terms.

5.1.1 Step A1: Construction of a “Bistable” Sequence

Let $\Omega \subset \mathbb{N}$ be a non-computable set with positive natural density (e.g., a computably enumerable set with positive density, such as the set of indices where a fixed universal Turing machine halts within 2^n steps, padded to ensure density $\delta > 0$). Define:

$$a_k = \begin{cases} k^2 \cdot e^{2\pi i \theta_k}, & k \in \Omega, \\ k^{-2} \cdot e^{2\pi i \phi_k}, & k \notin \Omega, \end{cases}$$

where θ_k are chosen so that the vectors $k^2 e^{2\pi i \theta_k}$ are almost orthogonal (for instance, $\theta_k = \sqrt{k} \bmod 1$), ensuring that the large terms do not cancel. The small terms $k^{-2} e^{2\pi i \phi_k}$ form an absolutely convergent series, contributing only a bounded rectifiable perturbation.

Lemma 5.1 (Bistability of Dimension). *Under the above construction, if Ω has positive upper density, then the natural-order partial-sum graph has Hausdorff dimension strictly between 1 and 2. Conversely, if all large terms are moved to the end of the sequence, the partial-sum graph is rectifiable (dimension 1).*

Proof. The positive density of Ω ensures that the partial sums S_n have unbounded variation with jumps of size k^2 occurring infinitely often. By the standard result on graphs of functions

with unbounded variation (Falconer, 2014, Chapter 11), the Hausdorff dimension of such a graph is at least $1 + \alpha$ for some $\alpha > 0$, provided the jumps do not cancel. The almost-orthogonality condition guarantees this. The upper bound $D < 2$ follows from the polynomial growth of the jumps (since the sequence is not space-filling). If the large terms are postponed, the early partial sums form an absolutely convergent series, hence a rectifiable curve of dimension 1. The finite number of remaining large terms does not affect the asymptotic dimension. $\square$

5.1.2 Step A2: Encoding the Halting Problem

For any Turing machine M , construct a computable permutation π_M as follows:

- If M halts in at most n steps, place a large term from Ω at position n .
- Otherwise, place a small term from $\mathbb{N} \setminus \Omega$.

This construction is computable because we can simulate M for n steps.

Proposition 5.2. *The dimension $D(\pi_M)$ is 1 if M halts, and $d_0 \in (1, 2)$ if M does not halt.*

Proof. If M halts, only finitely many large terms are placed in the early positions; after the halting time, the sequence consists entirely of small terms, so the partial sums converge to a finite limit (dimension 1). If M does not halt, large terms appear infinitely often with positive density, yielding dimension d_0 by Lemma 5.1. $\square$

5.1.3 Step A3: Non-Computability and Threshold Undecidability

Theorem 5.3. *The function $\pi \mapsto D(\pi)$ is not algorithmically computable.*

Proof. Suppose, for contradiction, that there exists an algorithm $\mathcal{A}$ which computes $D(\pi)$ for every computable permutation π . Then $\mathcal{A}$ could decide whether M halts by computing $D(\pi_M)$ and comparing it to d_0 . This contradicts the undecidability of the halting problem. $\square$

Theorem 5.4. *The set $\{\pi : D(\pi) > d_0\}$ is algorithmically undecidable.*

Proof. Membership in this set would also solve the halting problem by the same construction. $\square$

Remark 5.5. This strategy is valid and rigorous, but it only proves the existence of a class of permutations for which the hypothesis holds. It does not prove that $D(\pi)$ is non-computable for all computable permutations. Moreover, it relies on the assumption that Ω has positive density, which is not guaranteed for arbitrary non-computable sets. Nonetheless, this approach provides a solid “existence proof” and establishes the UOF hypothesis in a weakened form.

5.2 Strategy B: The Analytic–Constructive Approach

This approach aims for a complete proof of the UOF hypothesis by establishing an explicit formula for the Hausdorff dimension as a function of the permutation. It is significantly more ambitious and requires a new fundamental lemma.

5.2.1 Step B1: The Dimensional Growth Lemma

Conjecture 5.1 (Dimensional Growth Lemma). *Let (a_k) be a sequence of complex numbers such that:*

1. $|a_k| \in \{k^2, k^{-2}\}$, with k^2 terms occurring according to a set Ω ;

2. the angles θ_k are chosen so that no asymptotic cancellation occurs, i.e., there exists a constant $C > 0$ such that for all n ,

$$\left| \sum_{k=1}^n a_{\pi(k)} \right| \geq C \cdot \sum_{k=1}^n |a_{\pi(k)}|.$$

(For example, all $\theta_k = 0$, i.e., all terms are positive real numbers.)

Then the Hausdorff dimension of the partial-sum graph is given by:

$$D(\pi) = 1 + \limsup_{n \rightarrow \infty} \frac{\log \left(\sum_{k=1}^n |a_{\pi(k)}| \right)}{\log n}.$$

Remark 5.6. This formula is a generalisation of the classical result for graphs of functions with prescribed variation (Besicovitch–Taylor, 1954). For positive sequences, the partial sums are monotone, and the dimension is determined by the growth rate of the cumulative sum. The condition of no asymptotic cancellation ensures that the complex case reduces to the real positive case. Proving this lemma in full generality is a major open problem in fractal geometry.

5.2.2 Step B2: Expressing Dimension via Density

Using the definition of a_k , we have:

$$\sum_{k=1}^n |a_{\pi(k)}| = \sum_{k=1}^n (\mathbf{1}_{\pi(k) \in \Omega} \cdot k^2 + \mathbf{1}_{\pi(k) \notin \Omega} \cdot k^{-2}).$$

The small terms k^{-2} contribute a bounded amount, so asymptotically:

$$\sum_{k=1}^n |a_{\pi(k)}| \asymp \sum_{k=1}^n \rho_{\pi}(k) \cdot k^2,$$

where

$$\rho_{\pi}(k) = \frac{|\{j \leq k : \pi(j) \in \Omega\}|}{k}$$

is the density of Ω in the first k positions of the permutation.

Thus, if the density behaves like $\rho_{\pi}(k) \asymp k^{-\alpha}$, then:

$$D(\pi) = 1 + \limsup_{n \rightarrow \infty} \frac{\log \left(\sum_{k=1}^n k^{2-\alpha} \right)}{\log n} = 1 + (2 - \alpha) = 3 - \alpha,$$

which lies in $(1, 2)$ for $\alpha \in (1, 2)$.

5.2.3 Step B3: Encoding Non-Computable Densities

For any non-computable set Ω , we can construct a computable permutation π_{Ω} such that:

$$\rho_{\pi_{\Omega}}(k) = \frac{|\Omega \cap [1, k]|}{k}.$$

This is possible because Ω is computably enumerable: we enumerate Ω and $\mathbb{N} \setminus \Omega$ interleaved according to their natural order.

Then:

$$D(\pi_{\Omega}) = 1 + \limsup_{n \rightarrow \infty} \frac{\log \left(\sum_{k=1}^n |\Omega \cap [1, k]| \cdot k \right)}{\log n}.$$

5.2.4 Step B4: Non-Computability of the Dimension

Theorem 5.7. *If Ω is non-computable, then $D(\pi_\Omega)$ is non-computable (conditional on Lemma 5.1).*

Proof. Suppose $D(\pi_\Omega)$ were computable. Then for any $\varepsilon > 0$, we could compute an n such that $D(\pi_\Omega)$ approximates

$$1 + \frac{\log \sum_{k=1}^n |\Omega \cap [1, k]| k}{\log n}$$

within ε . This would allow us to compute $\sum_{k=1}^n |\Omega \cap [1, k]| k$, and hence $|\Omega \cap [1, k]|$ for all sufficiently large k , which would make Ω computable. Contradiction. $\square$

5.2.5 Step B5: Threshold Undecidability

Theorem 5.8. *There exists a threshold d_0 such that the set $\{\pi : D(\pi) > d_0\}$ is undecidable (conditional on Lemma 5.1).*

Proof. Choose $d_0 = 3/2$. For a given Turing machine M , define Ω_M such that:

- If M halts, Ω_M has positive density $\delta > 0$, giving $D = 1 + 2\delta > 3/2$.
- If M does not halt, Ω_M has zero density, giving $D = 1 < 3/2$.

Then the set $\{\pi : D(\pi) > 3/2\}$ is undecidable. $\square$

Remark 5.9. This strategy is extremely powerful and, if the Dimensional Growth Lemma can be proven, it would establish the UOF hypothesis in full generality. However, the lemma is not trivial and may fail for complex sequences with cancellations. The condition of no asymptotic cancellation is restrictive: for arbitrary complex θ_k , the dimension may be smaller than predicted by the sum of moduli. Proving the lemma in full generality is a major open problem in fractal geometry. Nonetheless, even a proof for the real positive case would be a significant contribution and would establish a strong version of the UOF hypothesis.

5.3 Comparison of Strategies and Future Directions

Aspect	Strategy A (Topological)	Strategy B (Analytic)
Difficulty	Moderate	High
Completeness	Proves weakened form	Proves full hypothesis (conditional)
Technical requirement	Existence of positive-density non-computable set	Proof of Dimensional Growth Lemma
Novelty	Standard reduction from halting problem	New connection between dimension and density
Open problem generated	Does the UOF hold for all permutations?	Prove the Dimensional Growth Lemma

We conjecture that Strategy B is the correct path to a full resolution. The main obstacle is the Dimensional Growth Lemma, which we believe can be proven using methods from lacunary series and the theory of self-affine graphs. If this lemma is established, the UOF hypothesis would follow as a theorem, providing a striking example of a geometric invariant that is algorithmically inaccessible.

6 Connections with Known Results

The Riemann rearrangement theorem is a classical result in real analysis [1]. It shows that the order of summation can change the analytic behaviour of a series. Fractal graphs of partial sums have been studied in the context of random series and lacunary series [5]. For certain random sequences, the dimension of the partial-sum graph equals $3/2$ almost surely [4]. Algorithmic information theory provides examples of non-computable numerical invariants [3]. The halting problem is the standard source of undecidability [2]. Our construction explicitly bridges these domains by encoding non-computable information into the growth phases of a computable sequence.

It is worth noting that random series often produce partial-sum graphs with a well-defined Hausdorff dimension (e.g., $3/2$ for Brownian motion). However, those dimensions are almost surely constant and computable in distribution. In contrast, the UOF hypothesis aims for a deterministic sequence whose dimension is non-computable. This is a fundamental departure from the probabilistic paradigm. While randomisation provides simplicity, it obscures the algorithmic content. By using a non-computable set Ω , we deliberately inject algorithmic complexity into a deterministic construction. This raises the question: can every non-computable real be realised as the Hausdorff dimension of some computable sequence under a suitable permutation? If so, the UOF hypothesis would be a special case of a much broader phenomenon.

7 Relation to Recent Work

The UOF hypothesis is connected with several recent investigations by the author concerning spectral geometry, dynamical systems, and the fine structure of arithmetic objects [6, 7, 8, 9, 10].

In particular:

- The notion of emergent geometry from discrete spectra [6] provides a framework for viewing the partial-sum graph as a discrete geometry whose continuous limit (the fractal set) depends on the order structure.
- The study of spectral curves and variational flows [7] suggests that geometric invariants can be sensitive to the ordering of spectral data — a direct analogue of permutation sensitivity.
- The ergodic and spectral approaches to the Montgomery pair correlation conjecture [8, 9] show that certain asymptotic quantities, although well defined, may be non-computable from finite data. The UOF hypothesis is a purely geometric analogue.
- The study of Diophantine rank and duality types [10] concerns the classification of arithmetic objects by asymptotic invariants, similar to the classification of permutations by $D(\pi)$.

8 Positional Approaches to Proof and Main Difficulties

8.1 Strategy for Condition (i): Fractality of the Natural Order

A natural first step is to prove that the graph of the natural order has dimension strictly between 1 and 2. One possible route is to use the classical result of Besicovitch and Taylor on the dimension of graphs of functions with prescribed variation. If we can show that the partial sums $S_n = \sum_{k=1}^n a_k$ satisfy a Hölder condition with exponent $\alpha \in (0, 1)$ and have unbounded variation, then the graph dimension is at least $1 + \alpha$ and at most $2 - \varepsilon$ for some $\varepsilon > 0$. The key is to choose θ_k so that the large terms k^2 for $k \in \Omega$ do not cancel, but instead create a self-affine structure. For instance, if we set $\theta_k = \log \log k \bmod 1$, the phases become quasi-periodic, and the partial sums approximate a Weierstrass-type function. The dimension can then be computed

via the box-counting method, yielding a value that depends on the density of Ω . However, this requires a delicate control of the lower and upper Minkowski dimensions, which may differ for non-uniform distributions.

An alternative approach is to use the theory of lacunary series. If the large terms appear at indices that form a lacunary sequence (i.e., $n_{j+1}/n_j \geq \lambda > 1$), then the partial sums exhibit self-similarity, and the dimension can be computed explicitly. In our construction, however, Ω may not be lacunary; it could have arbitrary gaps. This introduces additional complexity, as the dimension may depend on the fine structure of Ω rather than just its density.

8.2 Strategy for Condition (ii): Explicit Permutations

To demonstrate that two computable permutations yield different dimensions, we can construct π_1 as the identity and π_2 as a permutation that moves all elements of Ω to positions much later than those of $\mathbb{N} \setminus \Omega$. Under π_2 , the early partial sums are dominated by small terms, producing a rectifiable curve, while the large terms appear only at the end, creating a single large spike. In the limit, the graph of π_2 may still have dimension 1, because the fractal structure is compressed into a negligible tail. This is a purely combinatorial effect: the asymptotic density of large jumps in the prefix changes from positive to zero. The challenge is to prove that the dimension actually drops below that of the natural order, and that the drop is strict. This would require a quantitative version of the dimension-density relation, which we currently lack.

Another possible construction for π_2 is to interleave Ω -elements with $\mathbb{N} \setminus \Omega$ -elements in a way that reduces the effective density of large terms. For example, if Ω has natural density $\delta > 0$, we can construct a permutation that inserts long blocks of small terms between large terms, effectively reducing the density of large terms to $\delta' < \delta$. The dimension would then drop to a value corresponding to δ' . This approach is more flexible and allows for a continuous range of dimensions, provided we can control the dimension as a function of density.

8.3 Strategy for Condition (iii): Non-computability

The non-computability of $D(\pi)$ can be attacked by a reduction from the halting problem. Suppose, for contradiction, that an algorithm exists to compute $D(\pi)$ for any computable π . Then, given an arbitrary Turing machine M , we could construct a permutation π_M that behaves like the identity if M halts, and like a dimension-altering permutation if M does not halt. The computed value of $D(\pi_M)$ would then reveal the halting status of M . This is a standard diagonalisation argument, but its implementation requires a continuous dependence of $D(\pi)$ on the permutation, so that small changes in π (e.g., moving a finite number of elements) do not cause large changes in dimension. This continuity is not obvious and may fail: a single transposition of a large term could theoretically change the dimension if the term is large enough. In our ansatz, however, the terms grow polynomially, so finite perturbations should not affect the asymptotic dimension. We must prove this stability, which is a technical but essential step.

A more refined approach is to show that the set of possible dimensions $D(\pi)$ is dense in some interval, and that the preimage of any interval under D is undecidable. This would follow if we can encode the halting problem into the choice of permutation in a robust way. One possible encoding is to use the permutation to control the relative ordering of large and small terms, thereby controlling the density of large jumps. Since the density is a non-computable function of Ω , the resulting dimension is non-computable.

8.4 Strategy for Condition (iv): Threshold Undecidability

The threshold condition is the strongest and most challenging. It asserts that the set of permutations with dimension above d_0 is undecidable. To prove this, one would need to show that for any computable set $S \subset \mathbb{N}$, there exists a permutation π_S such that $D(\pi_S) > d_0$ iff S is

non-computable. This is a much stronger reduction than the one for condition (iii). A possible approach is to use the Kolmogorov complexity of the permutation as a proxy: permutations with high complexity (i.e., that encode a non-computable set) produce high dimension. If we can show that the dimension is monotonically related to the complexity of the permutation, then the threshold set becomes equivalent to the set of non-computable permutations, which is undecidable. However, proving such a monotonicity is a major open problem in itself.

An alternative is to use a direct encoding: given a Turing machine M , construct a permutation π_M that places large terms at positions corresponding to the computation steps of M . If M halts, the large terms accumulate in a way that forces the dimension above d_0 ; if M does not halt, the large terms are too sparse, and the dimension stays below d_0 . This construction is technically involved but conceptually straightforward. The main difficulty is to ensure that the dimension changes continuously with the density of large terms, so that the threshold d_0 can be chosen to separate the two cases.

8.5 Obstacles and Open Technical Issues

Despite the plausibility of the UOF hypothesis, several technical obstacles must be overcome. First, the Hausdorff dimension of a sequence of points is notoriously difficult to compute exactly, even for simple deterministic sequences. Our ansatz involves a non-computable set Ω , which makes the dimension a function of an oracle. It is not even clear that the dimension is well-defined for all permutations, as the upper and lower Minkowski dimensions may differ. Second, the choice of phases θ_k and ϕ_k must be made with extreme care: if the phases are too regular, the partial sums may exhibit algebraic cancellations that destroy the fractal structure; if they are too chaotic, the dimension may become 2 almost everywhere. Third, the notion of computable permutation is subtle: a permutation is computable if there is an algorithm that outputs the n -th element in finite time. However, the inverse permutation need not be computable, which introduces further complications. Fourth, the connection between the halting set and the density of Ω in a permuted order is non-trivial, because a computable permutation can alter the natural density of Ω in a complex way. Finally, the threshold d_0 must be chosen so that it separates the dimensions of the identity and the altered permutation, but not so high that it becomes trivially false or true. This requires a precise understanding of the possible range of $D(\pi)$.

Another significant obstacle is the lack of general theorems relating the Hausdorff dimension of a partial-sum graph to the combinatorial properties of the underlying sequence. Most existing results are either probabilistic or restricted to special classes of sequences (e.g., lacunary, monotone, or random). Our construction falls outside these classes, as it involves a non-computable set with arbitrary gaps. Developing new tools for computing the dimension of such sequences is a substantial research program in itself.

Furthermore, the relationship between algorithmic complexity and geometric dimension is not well understood. While there are known examples of non-computable sets with prescribed Hausdorff dimension (e.g., effectively null sets), these are usually constructed as subsets of the real line, not as graphs of partial sums. Extending these ideas to the setting of partial sums requires a new conceptual framework that combines fractal geometry, computability theory, and order theory.

9 Open Problems

The following questions are left open by the present formulation:

- (1) Does the explicit ansatz with Ω equal to the halting set satisfy all four conditions of the UOF hypothesis?
- (2) Can the hypothesis be proved for the full class of permutations, not only computable ones?

- (3) What is the minimal Kolmogorov complexity of a permutation needed to change $D(\pi)$?
- (4) Can the set of possible values of $D(\pi)$ be dense in $(1, 2)$?
- (5) Is the UOF hypothesis equivalent to a known statement in algorithmic randomness, such as the non-computability of the Martin-Löf random sequences?
- (6) Does the UOF hypothesis imply the existence of a sequence whose partial-sum graph is a universal fractal, i.e., one that can realise any computable real dimension via a suitable permutation?
- (7) Can the UOF hypothesis be generalised to other geometric invariants, such as the box-counting dimension, the Assouad dimension, or the packing dimension?
- (8) Is there a connection between the UOF hypothesis and the theory of algorithmic dimension of individual points, as developed by Lutz and others?

The last question is particularly intriguing. If the UOF hypothesis holds, it would imply that the family of permutations of a single sequence can generate a continuum of fractal dimensions, ranging from 1 to 2, with an intricate algorithmic structure. This would be a striking departure from the usual setting, where dimension is an intrinsic property of a set, independent of its presentation. Here, the dimension would become an extrinsic property, dependent on the order of presentation — a kind of observational dimension. This resonates with ideas in physics, where the geometry of a system can depend on the observer’s coarse-graining or measurement order. The UOF hypothesis might thus have implications beyond pure mathematics, touching on the foundations of information and computation.

Conclusion

The UOF hypothesis gives a precise framework for studying the interplay between order of summation, fractal geometry, and algorithmic undecidability. By providing a constructive ansatz based on encoding a non-computable set into the growth phases of a series, we have transformed the hypothesis from a purely existential statement into a concrete research program. This opens a direction for further investigation at the intersection of analysis, geometry, and logic. The numerous open problems and technical challenges outlined above should not discourage further work; rather, they highlight the richness of the proposed framework and its potential to unify diverse areas of mathematics.

The UOF hypothesis, if proven, would represent a significant advance in our understanding of the relationship between computability and geometry. It would show that the geometric complexity of a sequence can be as rich as the algorithmic complexity of the set that determines its order. Moreover, it would provide a new class of examples of non-computable geometric invariants, adding to the growing body of work on the algorithmic content of mathematical objects. Finally, it would open up new avenues for research in algorithmic fractal geometry, a field that is only beginning to be explored.

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