

From Persistent Filamentary Phases to Full Renormalization-Group Universality in Universal Modular Dynamics

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October 06, 2026

Abstract

Persistent infinite-volume filamentary structure is a geometric statement and, by itself, does not determine a complete dynamical universality class. We make this distinction precise for Universal Modular Dynamics (UMD) and resolve the associated T5 proof-or-counterexample problem. First, we show that the unrestricted implication from persistence of an asymptotic UMD filamentary phase to full renormalization-group (RG) universality-class equivalence is false: conservation laws, equilibrium structure, long-range interactions, absorbing-state data, nonunique RG limits, or nondecoupling spectator sectors can preserve the carrier while changing the critical theory. We then prove a restricted positive theorem for admissible UMD thermodynamic sequences whose active critical field is scalar, $\mathbb{Z}_2$ -odd, nonconserved, short-ranged, equilibrium and fluctuation–dissipation compatible, with codimension-one critical tuning in spatial dimension $d > 4$, unique full-sequence RG attraction, and either no spectator sector or a spectator satisfying the stated canonical quotient conditions. For this class, a common outcome-blind coarse-graining map, equality of the normalized linear and nonlinear RG data, control of the dangerously irrelevant quartic field, and bidirectional scaling-field and observable dictionaries yield

$$M/\sim_{\text{sp}} \simeq_{\text{RG}} A_{\text{ref}}(d), \quad d > 4,$$

where $A_{\text{ref}}(d)$ is the external scalar Gaussian Model-A reference. In particular, the quotient theories share the Gaussian Ornstein–Uhlenbeck fixed point, the normalized relevant and irrelevant spectrum, the nonlinear scaling-field normal form, and the quotient observable algebra with scale-decaying reconstruction error. The resulting exponents are $\eta = 0$, $\nu = 1/2$, and $z = 2$, while the quartic eigenvalue $y_u = 4 - d < 0$ is retained with its dangerous-irrelevance qualification for zero-mode and finite-size thermodynamic observables. This establishes the restricted theorem $\text{T5}_{\text{AG}, d>4}^+$ without promoting filamentary persistence into a universal class selector. The accompanying failure alternatives and spectator-indexed classification supply the first certified entry and the strict interface for a future UMD Universality Atlas.

Keywords: Universal Modular Dynamics; filamentary phase; renormalization group; universality class; Gaussian Model A; constructive ϕ^4 theory; spectator sectors.

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1 Introduction: From Filamentary Persistence to Full RG Universality

1.1 From a persistent carrier to a universality question

Universal Modular Dynamics (UMD) is formulated as a thermodynamic programme in which finite modular quantum systems, their quasi-local open-system evolution, and geometry extracted from modular observables are studied along admissible sequences of increasing size. The preceding stages of this programme established two logically different results. First, a finite-range percolative carrier was constructed in an external Poisson OR-5NN comparison geometry [31]. Second, under a restricted class of uniformly quasi-local UMD sequences, this carrier was transferred to an infinite-volume filamentary phase with persistent crossing and marked-observable content [32]. Those results solve the existence and persistence problem for the geometric carrier. They do not, however, identify the complete critical theory supported by that carrier.

This distinction is essential. Universality in the renormalization-group (RG) sense is not equality of pictures, connectivity events, or a small list of fitted exponents. It concerns attraction to a common fixed point together with the relevant and marginal spectrum, nonlinear scaling fields, dangerous variables, normalized observable content, and a controlled basin of attraction [5, 6, 9]. For dynamical critical phenomena, the classification also depends on conservation laws, reversible mode couplings, fluctuation–dissipation structure, and additional slow fields [1–4]. A filamentary carrier can survive changes in any of these data. Consequently, the passage from a persistent carrier to a full universality class requires a new theorem rather than a reinterpretation of the carrier theorem.

The present article addresses precisely that missing implication. We call it the *T5 problem*: determine whether an asymptotically persistent UMD filamentary phase forces equivalence with an external RG universality class, and, if not, identify a maximal explicitly certified subclass on which such an equivalence can be proved.

1.2 The unrestricted T5 statement and its obstruction

The strongest possible formulation would be

$$\begin{array}{ccc} \text{persistent asymptotic UMD filamentary phase} & & \\ \implies & & (1.1) \\ \text{full external universality-class equivalence.} & & \end{array}$$

Equation (1.1) is false without further hypotheses. The premise constrains a geometric and marked-observable sector, but it does not determine all slow degrees of freedom. In particular, one may preserve the same carrier while introducing a conserved scalar, an additional massless spectator, a protected topological sector, fluctuation–dissipation breaking, a long-range memory kernel, or several subsequential RG limits. Each modification can leave the carrier observable unchanged while altering the fixed point, dynamic exponent, relevant multiplicities, or observable algebra.

This failure is not merely a warning that assumptions might be incomplete. It is an identifiability obstruction. If two microscopic sequences agree on all filamentary observables but differ in an RG invariant that survives every admissible quotient, then no selector based only on filamentary persistence can assign both sequences to a unique external class. The correct

unrestricted verdict is therefore

$$\boxed{\text{persistent filamentary phase} \not\Rightarrow \text{full RG-universality equivalence.}} \quad (1.2)$$

This negative boundary is retained throughout the article. It is not weakened after a positive subclass is identified.

1.3 Comparison class and external reference

To obtain a positive result, we freeze a comparison class before using any critical outcome. Its elements are admissible UMD thermodynamic sequences $M = (M_N)_{N \geq 1}$ with a common thermodynamic embedding, uniformly quasi-local microscopic generators, and a persistent carrier satisfying the earlier T1–T4 contracts. The active slow field is required to be scalar, $\mathbb{Z}_2$ -odd, nonconserved, short-ranged, equilibrium and fluctuation–dissipation compatible. Criticality is reached by a codimension-one tuning, and the spatial dimension satisfies $d > 4$. Any spectator sector must either be absent or satisfy the explicit decoupling and quotient conditions stated below. Changing the symmetry, conservation law, noise rank, range class, absorbing-state structure, or nonquotient spectator content defines another comparison branch rather than an exception inside the theorem.

The external target is selected from the same outcome-blind fingerprint. For the certified branch it is the scalar equilibrium Gaussian Model-A reference, denoted by $A_{\text{ref}}(d)$. Model A is used here in the standard dynamic-critical sense of a nonconserved order parameter with relaxational equilibrium dynamics [1, 4]. It is not chosen because numerical exponents happen to agree. The reference is frozen before the UMD/reference equivalence test, and the equivalence is proved by common RG coordinates and observable dictionaries.

At the microscopic level, the UMD evolution is expressed through uniformly controlled completely positive Markov generators, in the structural class introduced by Gorini–Kossakowski–Sudarshan and Lindblad [21, 22]. Quasi-local propagation and thermodynamic compatibility are controlled using the same type of locality principles that underlie Lieb–Robinson bounds [24, 25]. These inputs make the limiting problem well posed; they do not themselves imply critical universality.

1.4 Main positive result

The principal theorem of this article is the following restricted positive T5 statement. Let M belong to the frozen comparison class and satisfy the full RG1–RG7 certificate: a common outcome-blind coarse-graining map; unique full-sequence attraction to the common fixed point; equality of the linearized spectrum; nonlinear scaling-field conjugacy with dangerous-irrelevance control; a valid spectator quotient; a unique noncolliding UMD subclass; and a bidirectional UMD/reference observable dictionary with reconstruction bounds. Then

$$\boxed{M / \sim_{\text{sp}} \simeq_{\text{RG}} A_{\text{ref}}(d), \quad d > 4.} \quad (1.3)$$

We denote (1.3) by

$$\boxed{\text{T5}_{A_G, d > 4}^+} \quad (1.4)$$

The relation $\simeq_{\text{RG}}$ is stronger than agreement of critical exponents and weaker than microscopic identity. It identifies the normalized fixed point, the relevant–marginal–irrelevant splitting, the

nonlinear scaling-field normal form, the dangerous-variable ledger, and the quotient observable algebra, modulo admissible analytic coordinate changes and certified spectator removal. The construction also provides inverse dictionaries on the quotient observable sector, with reconstruction error decaying under repeated coarse graining.

For the certified scalar branch, both sides approach the Gaussian Ornstein–Uhlenbeck fixed point. The leading critical data are

$$\eta = 0, \quad \nu = \frac{1}{2}, \quad z = 2, \quad y_u = 4 - d < 0. \quad (1.5)$$

The quartic coordinate is irrelevant in the linearized spectrum for $d > 4$ but is dangerously irrelevant for specified zero-mode and finite-size thermodynamic observables. It is therefore retained in the blown-up scaling-field chart rather than deleted as an ordinary correction. This qualification is part of the equivalence theorem.

The static multiscale component uses finite-range and polymer RG technology of the kind developed in rigorous renormalisation-group constructions [14, 15]. A recent map-level construction for weakly coupled short- and long-range $|\varphi|^4$ models is used only within its explicitly stated scope [16]. None of these external results is cited as already proving the UMD theorem: the UMD-specific thermodynamic embedding, dynamic locality, spectator adjudication, and two-way observable reconstruction are separate proof objects assembled here.

1.5 Noncircular proof architecture

The proof is deliberately modular. Its logical backbone is

$$\begin{aligned} \text{carrier} &\longrightarrow \text{critical fingerprint} \longrightarrow \text{RG1} - \text{RG4} \longrightarrow \text{RG5 spectator quotient} \\ &\longrightarrow \text{RG6 subclass} \longrightarrow \text{RG7 external equivalence.} \end{aligned} \quad (1.6)$$

Each arrow carries information not contained in the preceding object. Filamentary persistence does not select the dynamic fingerprint. Fingerprint agreement does not prove fixed-point attraction. Equality of a linear spectrum does not construct nonlinear physical scaling fields. Agreement of the active sector does not authorize deletion of a spectator. Finally, an internal UMD subclassification is not yet an equivalence with an external theory. The separation in (1.6) prevents the desired universality conclusion from being hidden in an assumption.

The seven final gates are used as follows:

- RG1:** construct one common outcome-blind coarse-graining map and Banach topology for UMD and reference objects;
- RG2:** prove tightness, asymptotic compactness, and unique full-sequence attraction to the common fixed point;
- RG3:** identify the linearized relevant, marginal, dangerous, and irrelevant spectrum, including multiplicities and invariant subspaces;
- RG4:** construct the nonlinear scaling-field conjugacy and control dynamic, memory, polymer, and dangerous-irrelevance remainders;
- RG5:** remove only spectators satisfying the complete decoupling and reconstruction contract, while rejecting nondecoupling EXT sectors;

- RG6:** assign a unique fingerprint- and spectator-indexed UMD universality subclass and prove branch noncollision;
- RG7:** identify the fixed external reference and establish bidirectional scaling-field and observable dictionaries with reconstruction control.

All gates refer to one compatible construction. Results proved for different coarse grainings or different spectator quotients cannot be spliced together to obtain T5.

1.6 Claim hierarchy and domain boundary

The article separates four claim tiers. Tier I is the inherited infinite-volume carrier and T1–T4 persistence theorem. Tier II identifies the scalar, nonconserved, equilibrium critical fingerprint. Tier III establishes the internal RG signature and UMD subclass after the spectator quotient. Tier IV proves equivalence of that quotient subclass with the fixed external Gaussian Model-A reference. Only Tier IV is the positive T5 theorem.

The result does not cover $d = 4$ logarithmic flow, $d < 4$ interacting fixed points, conserved or reversible mode-coupled dynamics, absorbing-state branches, uncontrolled long-range kernels, nonequilibrium FDT-breaking noise, or critical, hydrodynamic, protected, and relevantly coupled spectator sectors. It also does not assert microscopic equivalence or a realization of every external Model-A system by a UMD carrier. These exclusions are theorem boundaries, not finite-size corrections.

The combination of the negative unrestricted statement (1.2) and the positive restricted theorem (1.3) resolves the original proof-or-counterexample programme. The unrestricted statement fails, while a maximal proof-relative subclass admits a genuine external RG-equivalence theorem.

1.7 Article roadmap

Section 2 freezes the T5 claim contract, comparison class, and external-reference selection rule. Section 3 constructs the critical UMD class and codimension-one tuning manifold. Section 4 defines the common outcome-blind RG transformation and topology. Section 5 proves tightness and full-sequence fixed-point attraction. Section 6 identifies the linearized RG spectrum, and Section 7 constructs the physical scaling fields and critical-exponent correspondence. Section 8 controls irrelevant directions, dynamic and polymer remainders, and finite-size errors. Section 9 builds the spectator registry and canonical quotient. Section 10 proves the UMD subclassification and branch-noncollision theorem. Section 11 establishes the external Gaussian Model-A equivalence. Section 12 assembles the final theorem, exact domain, and exhaustive failure alternatives. The Outlook, Discussion, and Conclusion give the extension programme, interpretation, and final claim hierarchy. Appendices A–C record proof-object traceability, exact state-space and dictionary semantics, reproducibility contracts, the formal RG-equivalence relation, the counterexample library, and the strict interface to a future UMD Universality Atlas.

2 T5 Claim Contract and External Reference-Class Selection

2.1 Logical objects and order of quantifiers

Let $d > 4$ be fixed. A proposed UMD thermodynamic sequence is denoted by

$$M = (M_N)_{N \geq 1}, \quad M_N = (\mathcal{A}_N, \omega_N, \mathcal{L}_N, \lambda_N, \mathcal{O}_N), \quad (2.1)$$

where $\mathcal{A}_N$ is the finite observable algebra, ω_N its state, $\mathcal{L}_N$ the microscopic generator, λ_N the controllable parameter vector, and $\mathcal{O}_N$ the frozen local-observable registry. The thermodynamic embedding, carrier extraction rule, admissible perturbation class, coarse-graining convention, and normalization convention are fixed before any critical outcome is inspected.

The order of quantifiers is part of the theorem contract. We first choose one admissible sequence class $\mathfrak{D}_0(d)$, one external catalogue $\mathfrak{C}_{\text{cat}}(d)$, one fingerprint map $\mathfrak{F}$, and one comparison architecture $\mathcal{K}$. Only then do we quantify over $M \in \mathfrak{D}_0(d)$ and evaluate its certificates. Symbolically,

$$\exists (\mathfrak{D}_0, \mathfrak{C}_{\text{cat}}, \mathfrak{F}, \mathcal{K}) \text{ fixed prospectively,} \quad \forall M \in \mathfrak{D}_0 : \text{Adj}_{\text{T5}}(M) \text{ is evaluated.} \quad (2.2)$$

The reverse order, in which $\mathcal{K}$ or the external reference is selected after observing the flow of M , is excluded. In particular, agreement of measured exponents cannot be used to choose the comparison class that is later claimed to explain that agreement.

Definition 2.1 (Outcome-blind construction). A construction is *outcome-blind* if its observable registry, tuning coordinates, blocks, boundary convention, normalization, quotient rule, tie-breaking rule, external catalogue, and stopping rule are measurable with respect to the pre-outcome specification and are unchanged after critical exponents, survival events, fixed-point labels, or reference-match outcomes become available.

Outcome blindness does not mean that the RG trajectory is ignored. It means that the trajectory is evaluated by a map and an adjudicator whose semantics were fixed independently of the desired conclusion.

2.2 Exact T5 claim contract

The unrestricted statement is the implication

$$\text{Carrier}_\infty(M) \implies M \simeq_{\text{RG}} \mathfrak{C}_{\text{ext}}. \quad (2.3)$$

As explained in Section 1, this statement is not identifiable from carrier observables and is false on the unrestricted UMD class. The positive target must therefore name both a domain and a quotient.

Let $\sim_{\text{sp}}$ remove only sectors that satisfy the complete spectator contract of Section 9. Let $\mathbf{A}_{\text{ref}}(d)$ denote the external scalar, short-range, equilibrium Gaussian Model-A reference selected below. The exact positive claim is

$$\text{T5}_{\mathbf{A}_G, d > 4}^+ : \quad \forall M \in \mathfrak{D}_{\text{T5}}^+(d), \quad M / \sim_{\text{sp}} \simeq_{\text{RG}} \mathbf{A}_{\text{ref}}(d). \quad (2.4)$$

Here $\simeq_{\text{RG}}$ denotes equality of the normalized RG signature up to admissible local analytic changes of scaling coordinates, redundant directions, and the certified spectator quotient. It

includes fixed-point attraction, invariant-subspace and spectrum matching, nonlinear scaling-field conjugacy, the dangerous-variable ledger, and bidirectional reconstruction on the quotient observable algebra. It does not mean equality of microscopic Hamiltonians, generators, metric factors, or configurations.

The claim is directional at the microscopic level. It proves that a member of the certified UMD class enters the external Model-A universality class. It does not assert that every external Model-A theory possesses a microscopic UMD realization or a filamentary carrier.

2.3 Frozen positive domain

The theorem domain is the intersection of fifteen independently auditable contracts,

$$\mathfrak{D}_{T5}^+(d) := \bigcap_{i=1}^{15} \mathfrak{D}(H_i), \quad d > 4. \quad (2.5)$$

They are grouped as follows.

ID	Frozen contract
H_1	The finite UMD objects admit the fixed thermodynamic embedding, uniform quasi-locality, and microscopic consistency bounds.
H_2	The outcome-blind filamentary carrier exists on the certified phase interval and satisfies the inherited T1–T4 persistence, crossing-density, and mark-stability conclusions.
H_3	Stabilization, local-weak transfer, block contiguity, and dependence-defect bounds transfer the carrier and its local observables to the thermodynamic limit.
H_4	The active slow sector contains one real scalar $\mathbb{Z}_2$ -odd field.
H_5	The active field is nonconserved and purely relaxational.
H_6	Detailed balance and fluctuation–dissipation compatibility hold, and there is no absorbing-state constraint.
H_7	The active kernel is short-ranged and the critical tuning manifold has codimension $r = 1$.
H_8	The spatial dimension satisfies $d > 4$.
H_9	RG1 provides one common outcome-blind UMD/reference coarse-graining map, Banach topology, and relevant-field chart.
H_{10}	RG2 gives unique full-sequence attraction to the common Gaussian Ornstein–Uhlenbeck fixed point.
H_{11}	RG3 gives equality of the relevant, marginal, dangerous, and irrelevant linearized spectra.
H_{12}	RG4 supplies nonlinear scaling-field conjugacy, dynamic locality, and control of the dangerously irrelevant quartic coordinate.
H_{13}	RG5 classifies the spectator sector as PURE or as QUOT satisfying all conditions S1–S7.
H_{14}	RG6 assigns the sequence uniquely to a noncolliding fingerprint- and spectator-indexed UMD subclass.
H_{15}	RG7 supplies the fixed external reference, two-way scaling-field and observable dictionaries, and quotient reconstruction bounds.

The assumptions are conjunctive. No weighted score, majority rule, or numerical closeness criterion can replace membership in $\mathfrak{D}_{T5}^+(d)$. A sequence that fails one contract may still be critical or universal in another sense, but it is outside the positive T5 domain defined here.

For later adjudication, define the nested domain lattice

$$\begin{aligned}\mathfrak{D}_1 &= \mathfrak{D}_0 \cap \mathfrak{D}(H_1), \\ \mathfrak{D}_2 &= \mathfrak{D}_1 \cap \mathfrak{D}(H_2, H_3), \\ \mathfrak{D}_3 &= \mathfrak{D}_2 \cap \mathfrak{D}(H_4, \dots, H_8), \\ \mathfrak{D}_4 &= \mathfrak{D}_3 \cap \mathfrak{D}(H_9, \dots, H_{12}), \\ \mathfrak{D}_5 &= \mathfrak{D}_4 \cap \mathfrak{D}(H_{13}, H_{14}), \\ \mathfrak{D}_6 &= \mathfrak{D}_5 \cap \mathfrak{D}(H_{15}) = \mathfrak{D}_{T5}^+(d).\end{aligned}\tag{2.6}$$

Thus $\mathfrak{D}_6 \subseteq \dots \subseteq \mathfrak{D}_1 \subseteq \mathfrak{D}_0$, and membership never propagates upward without the additional certificate. In particular,

$$M \in \mathfrak{D}_2 \not\Rightarrow M \in \mathfrak{D}_3.\tag{2.7}$$

2.4 Fingerprint and comparison-class freeze

The critical comparison class is indexed by the normalized fingerprint

$$\mathfrak{F}(M) = (d_{\text{eff}}, \Sigma, \mathcal{Q}, \mathcal{A}_{\text{abs}}, \mathcal{S}, \mathcal{N}, \mathcal{D}, \chi)_M.\tag{2.8}$$

The entries record effective dimension and anisotropy, faithful symmetry, conserved-charge algebra, absorbing stationary faces, slow-field bundle, noise rank, dissipative/reversible structure and equilibrium status, and surviving topological or superselection data. The carrier law belongs to the observable registry but is not substituted for this fingerprint.

Let $u = (u_1, \dots, u_r)$ be the preregistered relevant-coordinate map. Near a reference point p_* , the critical set is

$$\mathcal{M}_c = u^{-1}(0).\tag{2.9}$$

If u is C^k , $k \geq 2$, and

$$\inf_{p \in U} \sigma_{\min}(Du_p) \geq \kappa_{\text{tun}} > 0,\tag{2.10}$$

then 0 is a regular value and $\mathcal{M}_c \cap U$ is a codimension- r submanifold. The present positive branch has $r = 1$. One connected component with constant protected fingerprint defines the comparison class $\mathfrak{U}_*$. Closure of a protection gap, change of a discrete fingerprint datum, loss of transversality, or activation of a nonquotient spectator exits $\mathfrak{U}_*$; it does not create an exception inside it.

Definition 2.2 (Comparison-class freeze). The comparison class is frozen by the tuple

$$\mathfrak{C}_* = [\mathfrak{U}_*, \mathcal{O}_*, u, \mathfrak{F}_*, \mathcal{R}_b, Q_{\text{sp}}, N_b, \mathfrak{C}_{\text{cat}}].\tag{2.11}$$

No entry of this tuple may be changed in response to the observed RG orbit or to the success or failure of an external match.

2.5 Outcome-blind external-reference selection

Let the preregistered catalogue $\mathfrak{C}_{\text{cat}}$ contain external critical theories with normalized fingerprints computed in the same registry. The first-stage candidate set is

$$\mathfrak{C}_*(M) := \{C \in \mathfrak{C}_{\text{cat}} : \mathfrak{F}(C) \cong \mathfrak{F}(M)\}. \quad (2.12)$$

Selection obeys the following prospective rule:

- (i) if $|\mathfrak{C}_*(M)| = 0$, no known external reference is assigned;
- (ii) if $|\mathfrak{C}_*(M)| = 1$, its unique member is frozen as the candidate reference;
- (iii) if $|\mathfrak{C}_*(M)| > 1$, the candidates are separated only by a preregistered RG-spectrum or slow-mode discriminator; absence of such a discriminator yields an unresolved verdict.

Measured UMD exponents, preferred physical interpretation, and carrier appearance are not tie-breakers.

For the branch satisfying H_4 – H_8 , the selector returns the scalar short-range equilibrium Model-A candidate. Subsequent RG1–RG7 tests establish or reject equivalence; the fingerprint match alone does not establish it. This distinction preserves the standard dynamical role of symmetry, conservation, and equilibrium structure in the Model-A classification [1, 4].

2.6 Final RG1–RG7 success gate

The final gate semantics are fixed as follows.

Gate	Required proof object	Licensed conclusion
RG1	Common outcome-blind coarse-graining semigroup, Banach topology, operational embeddings, and relevant-field chart.	UMD and reference flows belong to one comparison category.
RG2	Tightness, tail control, asymptotic compactness, and unique full-sequence fixed-point attraction.	Both tuned sequences enter the same Gaussian Ornstein–Uhlenbeck basin.
RG3	Equality of the linearized RG operators on normalized quotient coordinates, including multiplicities and invariant subspaces.	Relevant, marginal, dangerous, and irrelevant spectra agree.
RG4	Nonlinear scaling-field conjugacy, quartic dangerous-irrelevance ledger, dynamic locality, and summable memory/polymer remainders.	The nonlinear normal forms and physical scaling coordinates agree.
RG5	Spectator registry and complete S1–S7 quotient certificate.	PURE and QUOT branches preserve the active signature; EXT branches are not deleted.

Gate	Required proof object	Licensed conclusion
RG6	Exhaustive, nonoverlapping fingerprint- and spectator-indexed subclass selector with branch noncollision.	A unique internal UMD universality subclass is assigned.
RG7	Frozen external reference, bidirectional scaling-field and observable dictionaries, and scale-decaying reconstruction bounds.	External Gaussian Model-A RG equivalence is established.

The success predicate is strictly conjunctive:

$$\text{PASS}_{\text{T5}}(M) = \bigwedge_{i=1}^{15} H_i(M). \quad (2.13)$$

All gates must refer to the same registry, block map, normalization, tuning chart, and spectator quotient. A collection of individually valid gates from incompatible constructions is not a T5 certificate.

2.7 Proof-or-counterexample decision rule

Define the ordered first-failure adjudicator

$$\text{Adj}_{\text{T5}}(M) \in \{F_0, F_1, \dots, F_8, P\}. \quad (2.14)$$

Evaluation stops at the first failed or unresolved contract.

Code	First failed layer	Scientific verdict
F_0	H_1	No admissible thermodynamic UMD object.
F_1	H_2-H_3	Infinite carrier persistence is not established.
F_2	H_4-H_8	The carrier may survive, but the frozen scalar Model-A fingerprint is absent.
F_3	$\text{RG1}/H_9$	No common outcome-blind RG comparison exists.
F_4	$\text{RG2}/H_{10}$	Unique full-sequence Gaussian attraction is not established.
F_5	$\text{RG3-RG4}/H_{11}-H_{12}$	The linear spectrum or nonlinear scaling-field structure differs or remains unresolved.
F_6	$\text{RG5}/H_{13}$	A critical, hydrodynamic, protected, or relevantly coupled spectator survives.
F_7	$\text{RG6}/H_{14}$	The internal subclass is different or unresolved.
F_8	$\text{RG7}/H_{15}$	An internal UMD subclass exists, but external equivalence is not certified.
P	none	All H_1-H_{15} pass; the restricted positive T5 theorem is licensed.

The ordered rule is exhaustive on the proposed domain and mutually exclusive by construction. A failure code is not automatically a proof of absence of criticality. It states that the particular

conclusion (2.4) is not licensed. Depending on the failed layer, the correct output can be a different fingerprint branch, a product or coupled EXT class, a runaway flow, a new universality subclass, or an unresolved case.

The project-level decision has three possible forms:

D1: Positive theorem. At least one nonempty frozen subclass satisfies H_1 – H_{15} and yields (2.4).

D2: Classification theorem. Several protected fingerprints or spectator signatures lead to distinct, noncolliding subclasses.

D3: Counterexample theorem. Two sequences share the persistent carrier premise while differing in an invariant required for RG equivalence, refuting (2.3).

The present article realizes all three layers coherently: it proves the restricted positive entry, constructs the subclass selector, and retains the counterexample boundary to the unrestricted implication.

2.8 Circularity firewall and contract theorem

The following operations are forbidden: selecting the external reference from measured exponent agreement; changing the observable registry after inspecting the RG orbit; discarding a spectator merely because it is invisible to the carrier; deleting the quartic coordinate before testing singular observables; combining gates proved under different normalizations; or interpreting a subsequence limit as full-sequence attraction.

Theorem 2.3 (T5 contract and adjudication). *Fix the tuple in Theorem 2.2, the outcome-blind semantics of Theorem 2.1, and the ordered contracts H_1 – H_{15} . Then every proposed sequence $M \in \mathfrak{D}_0(d)$ receives exactly one first-failure or success code under Adj_{T5} . The code P licenses the claim (2.4); every code F_i forbids that claim while preserving all lower-tier conclusions certified before the first failure. No adjudication outcome can alter the frozen selector, reference catalogue, coarse-graining map, or quotient rule.*

Proof. The contracts form a finite ordered list, so evaluation has a first nonpass element unless all contracts pass. Stopping at that element makes the codes mutually exclusive, and inclusion of the all-pass code makes the list exhaustive. By Theorem 2.1, every semantic ingredient of the adjudicator is fixed before the evaluated outcomes. Hence a failure cannot be repaired by retrospective reselection, and success cannot be created by conditioning on the desired external label. The implication associated with P is exactly (2.4); lower codes lack at least one necessary conjunct and therefore do not license it. $\square$

This theorem closes the formal contract layer. The remaining sections must construct the listed proof objects without modifying the domain, reference selector, gate meanings, or decision rule.

3 Critical UMD Model Class and Tuning Manifold

3.1 Ambient thermodynamic UMD space

Let $\mathfrak{P}$ be a separable Banach manifold of UMD thermodynamic data

$$p = (\mathcal{A}, \omega, \mathcal{L}, \lambda, \eta), \quad (3.1)$$

where $\mathcal{A}$ is a quasi-local observable net, ω a compatible state family, $\mathcal{L}$ a uniformly quasi-local generator family, $\lambda \in \mathbb{R}^m$ the controllable coupling vector, and η an admissible microscopic perturbation. A point p represents a sequence of finite systems along the frozen exhaustion, not an isolated finite-volume model. The topology on $\mathfrak{P}$ controls local states, local generator densities, their quasi-local tails, and the observables in the registry of Section 2.

Write $\mathfrak{P}_{\text{adm}} \subset \mathfrak{P}$ for the sequences satisfying H_1 – H_3 : the thermodynamic embedding exists, the local generators are uniformly controlled, and the previously certified carrier and its T1–T4 observables transfer to the limit. The inclusion of H_2 – H_3 is important: the critical class constructed below is the T5 comparison class, not the class of all scalar Model-A systems.

The microscopic generator is required to produce a completely positive, trace-preserving evolution and to be compatible with the thermodynamic embedding. In finite quantum volumes it has the GKSL structural form [21, 22]; in the commutative continuous-variable branch used below it is the Markov diffusion counterpart of that structure. Uniform quasi-locality is the property needed to pass from these finite generators to a consistent infinite-volume dynamics, in the same general locality setting in which Lieb–Robinson estimates control propagation [24, 25].

Definition 3.1 (Admissible critical UMD class). An admissible critical UMD sequence is a point $p \in \mathfrak{P}_{\text{adm}}$ lying on the preregistered critical tuning manifold, with a protected slow sector and a locally constant normalized fingerprint. The set of such points in one connected protected component is denoted by $\mathfrak{U}_*$. Membership in $\mathfrak{U}_*$ does not yet assert RG fixed-point attraction; that is RG2.

3.2 Relevant coordinates and codimension-one tuning

Let

$$u = (u_1, \dots, u_r) : \mathfrak{P}_{\text{adm}} \longrightarrow \mathbb{R}^r \quad (3.2)$$

be the relevant-coordinate map frozen by the common reference linearization. The critical set is

$$\mathcal{M}_c := u^{-1}(0). \quad (3.3)$$

The positive branch treated in this article has one thermal relevant coordinate, hence $r = 1$. We nevertheless state the geometric result for general finite r because branch splitting can change the number of relevant directions.

Assumption 3.2 (Uniform transversality). There exist an open neighbourhood $U \subset \mathfrak{P}_{\text{adm}}$, an integer $k \geq 2$, and $\kappa_{\text{tun}} > 0$ such that

$$u \in C^k(U, \mathbb{R}^r), \quad \inf_{p \in U} \sigma_{\min}(Du_p) \geq \kappa_{\text{tun}}. \quad (3.4)$$

Proposition 3.3 (Tuning-manifold theorem). *Under Theorem 3.2, 0 is a regular value of u*

and $\mathcal{M}_c \cap U$ is a C^k embedded Banach submanifold of codimension r . For every $p \in \mathcal{M}_c \cap U$,

$$T_p \mathcal{M}_c = \ker Du_p. \quad (3.5)$$

In the certified scalar branch, $r = 1$ and the tuning manifold is locally a codimension-one hypersurface.

Proof. The lower singular-value bound makes Du_p surjective with a uniformly complemented kernel. The Banach-space regular-value theorem then gives the submanifold and tangent-space statements. Codimension equals $\text{rank } Du_p = r$, which is constant on U . $\square$

The theorem distinguishes the exact thermal scaling field from a bare mass or temperature parameter. For a microscopic quartic model one may write

$$t = r - r_c(u, T, \kappa), \quad (3.6)$$

but r_c must be identified by the thermodynamic/RG construction. Replacing t by the bare coefficient r without proving (3.6) would not certify membership in $\mathcal{M}_c$.

3.3 Normalized critical fingerprint

For $p \in \mathcal{M}_c$, define the protected fingerprint

$$\mathfrak{F}(p) = (d_{\text{eff}}, \Sigma, \mathcal{Q}, \mathcal{A}_{\text{abs}}, \mathcal{S}, \mathcal{N}, \mathcal{D}, \chi)_p. \quad (3.7)$$

Here d_{eff} includes anisotropic weights; Σ is the faithful symmetry action on the slow sector; $\mathcal{Q}$ is the local/quasi-local conserved-charge algebra; $\mathcal{A}_{\text{abs}}$ is the incidence structure of absorbing stationary faces; $\mathcal{S}$ is the slow-field bundle; $\mathcal{N}$ records the noise rank and null space; $\mathcal{D}$ records dissipative and reversible couplings together with detailed balance or KMS/FDT status; and χ contains topological or superselection labels that survive the spectator quotient.

The positive branch is specified by

$$\mathfrak{F}_{AG} = (d, \mathbb{Z}_2, \{0\}, \emptyset, \mathbb{R}_{\text{odd}}, 1, \text{relaxational/FDT}, \emptyset), \quad d > 4. \quad (3.8)$$

Thus the active sector contains one real $\mathbb{Z}_2$ -odd scalar; its order parameter is not conserved; the noise is local and nondegenerate in that sector; there is no absorbing face; the dynamics is purely relaxational and equilibrium/FDT compatible; and no protected label survives the admissible quotient. These are precisely the fingerprint data of the scalar Model-A branch in the standard dynamic classification [1, 4]. The Gaussian suffix G is not inferred from the fingerprint alone; it is supplied later by RG2 fixed-point attraction in $d > 4$.

3.4 Protection gaps

Local constancy of (3.7) requires quantitative separation from competing branches. We freeze the following protection registry:

$$\Delta_*(p) := \min\{\kappa_{\text{tun}}, \Delta_{\text{slow}}, \Delta_{\Sigma}, \Delta_{\mathcal{Q}}, \Delta_{\text{abs}}, \Delta_{\mathcal{N}}, \Delta_{\text{range}}, \Delta_{\text{sp}}\}. \quad (3.9)$$

The terms have the following meanings:

- Δ_{slow} separates the selected slow Riesz projection from the fast generator spectrum;
- Δ_{Σ} protects the faithful symmetry representation and its multiplicities;
- $\Delta_{\mathcal{Q}}$ separates zero modes generated by exact charges from nonconserved slow relaxation;
- Δ_{abs} gives normal hyperbolicity of the stationary faces and protects the absence of an absorbing-state bifurcation;
- $\Delta_{\mathcal{N}}$ protects the rank and symmetry type of the effective noise kernel;
- Δ_{range} separates the short-range kernel from a relevant long-range tail;
- Δ_{sp} separates removable gapped spectators from critical, hydrodynamic, or protected EXT sectors.

The comparison class uses a uniform bound

$$\inf_{p \in C_*} \Delta_*(p) > 0 \quad (3.10)$$

on a connected component $C_* \subset \mathcal{M}_c \cap U$.

Theorem 3.4 (Fingerprint stability). *Assume Theorem 3.2, the uniform gap condition (3.10), norm-continuity of the slow Riesz projections, unitary equivalence of the symmetry action along C_* , and normal hyperbolicity of the stationary-face structure. Then the discrete entries of $\mathfrak{F}$ are constant on C_* , its bundle and kernel entries are continuously conjugate, and the normalized fingerprint has a single equivalence value $[\mathfrak{F}_*]$ throughout the comparison class.*

Proof. Uniform spectral separation makes the Riesz projections norm-continuous. Their finite ranks and representation multiplicities are integer-valued and therefore constant on a connected set. Exact charge rank and noise-kernel rank are constant by the same argument. Normal hyperbolicity preserves the stationary faces and their incidence relations. The short-range and spectator gaps prevent a change of range class or the appearance of an unregistered slow factor. Continuous conjugacies identify the remaining bundle data, yielding one normalized fingerprint class. $\square$

3.5 Branch-splitting theorem

Let $\gamma : [0, 1] \rightarrow \mathcal{M}_c$ be a continuous path of critical UMD data. Suppose $\mathfrak{F}(\gamma(0))$ and $\mathfrak{F}(\gamma(1))$ are not equivalent. The path cannot remain inside one protected comparison class.

Theorem 3.5 (Protected branch splitting). *If the endpoints of γ have inequivalent normalized fingerprints, then there exists $s_* \in [0, 1]$ at which at least one of the following occurs:*

- (a) $Du_{\gamma(s_*)}$ loses rank;
- (b) the slow/fast spectral gap closes;
- (c) the symmetry representation or its multiplicity changes;
- (d) a conserved charge appears or disappears;
- (e) an absorbing stationary face bifurcates;

(f) the effective noise rank changes;

(g) a long-range tail becomes relevant;

(h) a nonquotient spectator becomes slow, protected, or relevantly coupled.

Consequently, the two endpoints belong to different UMD critical subclasses. No finite list of exceptional points can be patched into one frozen comparison class without invalidating its fingerprint contract.

Proof. If no listed event occurred, all terms in (3.9) would remain positive along the compact path and the transversality rank would remain constant. The argument of Theorem 3.4 would then provide a continuous conjugacy of every fingerprint component from one endpoint to the other, contradicting their assumed inequivalence. $\square$

This theorem turns qualitative alternatives into exact class boundaries. A conserved scalar leads to a Model-B-type candidate; coupling the nonconserved field to a conserved density leads to a Model-C-type or coupled EXT candidate; an absorbing face produces an absorbing-state branch; and a critical spectator produces a product or coupled EXT fixed point. None is a failure of criticality, but none belongs to the frozen scalar Model-A comparison class.

3.6 Concrete microscopic representative

The scalar branch is nonempty at the microscopic modular level. Let $\Lambda \in \mathbb{Z}^d$, $d > 4$, and $\phi = (\phi_x)_{x \in \Lambda} \in \mathbb{R}^\Lambda$. For $\kappa, \Gamma, T, u > 0$, define

$$H_\Lambda(\phi; r, u, h) = \frac{\kappa}{2} \sum_{\langle x, y \rangle} (\phi_x - \phi_y)^2 + \sum_{x \in \Lambda} \left(\frac{r}{2} \phi_x^2 + \frac{u}{4!} \phi_x^4 - h \phi_x \right). \quad (3.11)$$

The finite-volume Gibbs law

$$\mu_\Lambda(d\phi) = Z_\Lambda^{-1} \exp[-H_\Lambda(\phi; r, u, h)/T] d\phi \quad (3.12)$$

is well defined by quartic coercivity. On smooth cylinder observables, set

$$G_\Lambda f = \Gamma T \sum_{x \in \Lambda} \partial_{\phi_x}^2 f - \Gamma \sum_{x \in \Lambda} (\partial_{\phi_x} H_\Lambda) \partial_{\phi_x} f. \quad (3.13)$$

This is a finite-range, globally well-posed, reversible Model-A diffusion. Its Dirichlet form is

$$- \int f G_\Lambda g d\mu_\Lambda = \Gamma T \sum_x \int (\partial_x f)(\partial_x g) d\mu_\Lambda. \quad (3.14)$$

Thus the invariant measure, detailed balance, nonconservation of the order parameter, local noise, and absence of an absorbing configuration are exact microscopic properties rather than assumptions imposed on an effective action.

At $h = 0$, the transformation $\phi \mapsto -\phi$ gives the faithful $\mathbb{Z}_2$ symmetry. The normalized dangerous-variable coordinates

$$\psi = \sqrt{u} \phi, \quad H = \sqrt{u} h, \quad \Theta = uT \quad (3.15)$$

conjugate the finite-volume dynamics to a quartic Langevin system with noise strength Θ . These coordinates retain the reconstruction $\phi = u^{-1/2}\psi$ and prevent the quartic field from being discarded when it is RG-irrelevant but thermodynamically dangerous.

The Gaussian case $u = 0$, understood with positive mass followed by the ordered critical limit, supplies an exactly solvable Ornstein–Uhlenbeck benchmark. For $u > 0$, (3.11) supplies the interacting microscopic parent whose tuned critical trajectory is constructed in later sections using rigorous multiscale RG methods

3.7 Section theorem and scope

Theorem 3.6 (Critical scalar UMD comparison class). *Assume H_1 – H_8 , Theorem 3.2, and the uniform protection gaps (3.10). Then the critical set contains a protected codimension-one component C_* on which the normalized fingerprint is constant and equal to (3.8). The corresponding sequences form one scalar, nonconserved, short-range, equilibrium/FDT UMD comparison class $\mathfrak{U}_*$. Every continuous exit from this class crosses one of the certified branch boundaries in Theorem 3.5.*

The theorem closes the class-construction and tuning layer. It does not infer Gaussian attraction from $d > 4$, does not delete spectator sectors, and does not yet establish external equivalence. Those conclusions require RG1–RG7 in Sections 4–11..

4 Common Outcome-Blind Coarse-Graining Transformation

4.1 Common operational hierarchy

The microscopic UMD object and the external stochastic reference do not live in the same native state space. A direct comparison of their microscopic coordinates would therefore be representation dependent. We instead compare their operational response and correlation hierarchies in one common space.

Fix prospectively a countable separating registry of bounded local fields

$$\mathbf{O} = (O_a)_{a \in \mathbb{N}}, \quad (4.1)$$

together with their response partners, space–time chart, source convention, and normalization. For a UMD thermodynamic object p , let $Z_p[J_+, J_-]$ be the Schwinger–Keldysh/operational generating functional and write its connected hierarchy as

$$G_{p, a_1 \dots a_n}^{(n)} = \left. \frac{\delta^n \log Z_p[J]}{\delta J_{a_1} \dots \delta J_{a_n}} \right|_{J=0}. \quad (4.2)$$

For the classical Model-A reference q , the MSRJD generating functional is defined with the same registry labels and source normalization [2–4]. Thus

$$\iota_U : p \longmapsto \mathbf{G}_p, \quad \iota_R : q \longmapsto \mathbf{G}_q \quad (4.3)$$

have one codomain. Equation (4.3) assumes neither equality of the hierarchies nor universality; it supplies only common operational type.

The registry is separating on the frozen quotient: if all registered connected correlation and response functionals agree, the two quotient operational states agree. This condition is essential

for later reconstruction. A nonseparating registry could conceal a relevant spectator and is therefore an RG1 failure.

4.2 Weighted Banach hierarchy topology

For a connected n -point kernel supported on a space-time configuration $X = (x_1, t_1; \dots; x_n, t_n)$, let $\ell_T(X)$ be the minimal tree length of its support and let $w_a \geq 1$ be the frozen field weights. Define

$$\|\mathbf{G}\|_{\mu, s, \rho} := \sum_{n \geq 1} \frac{\rho^n}{n!} \sup_{a_1, \dots, a_n} \frac{1}{\prod_{j=1}^n w_{a_j}} \sup_X \left[e^{\mu \ell_T(X)} (1 + \text{diam } X)^s \left| G_{a_1 \dots a_n}^{(n)}(X) \right| \right]. \quad (4.4)$$

Let $\mathbb{B}_{\mu, s, \rho}$ be the completion of normalized realizable hierarchies under (4.4), with causality, Hermiticity/reality, symmetry, and applicable Ward identities imposed as a closed subspace. The factorial weight controls the all-order registry, the tree weight measures locality, and the polynomial factor controls residual infrared growth.

Blocking need not preserve the strongest norm. We therefore freeze a weaker target space

$$\mathbb{B}_- := \mathbb{B}_{\mu', s', \rho'}, \quad 0 < \mu' < \mu, \quad s' \leq s, \quad 0 < \rho' < \rho. \quad (4.5)$$

The loss from $\mathbb{B}_+$ to $\mathbb{B}_-$ is fixed in advance and cannot be enlarged after observing a difficult trajectory. This two-topology structure separates compactness and differentiability requirements without truncating the hierarchy.

4.3 Exact outcome-blind block semigroup

Fix an integer block factor $b_0 \geq 2$, nested half-open spatial blocks, a deterministic boundary convention, a compatible time mesh, a dynamic-rescaling symbol z , and a field-dimension matrix $D(b)$. For $b = b_0^n$, define the blocked field schematically by

$$(B_b \Phi)_a(X, T) = b^{\Delta_a} \sum_c D_{ac}(b) \frac{1}{|Q_b(X)|} \sum_{x \in Q_b(X)} \frac{1}{|I_{b^z}(T)|} \int_{I_{b^z}(T)} \Phi_c(x, t) dt. \quad (4.6)$$

For noninteger b^z , temporal blocking is defined by the frozen spectral interpolation or by a compatible nested time partition. Mixing these conventions is not allowed.

The kernels and field rescalings satisfy

$$B_{b_0^m} B_{b_0^n} = B_{b_0^{m+n}}, \quad D(bc) = D(b)D(c). \quad (4.7)$$

Let $(B_b)_\#$ denote pushforward of the complete connected hierarchy. No cumulant order is discarded. This is an exact Wilsonian comparison map in operational coordinates, consistent with the principle that coarse graining is fixed before the effective truncation is justified [5, 11, 13].

Let Q_{sp} be the prospectively defined redundancy/spectator quotient and N_b the frozen centering and two-point normalization. They obey

$$Q_{\text{sp}} B_b = B_b Q_{\text{sp}}, \quad N_{bc} = N_b \circ (B_b)_\# \circ N_c \quad \text{on normalized representatives.} \quad (4.8)$$

The exact RG map is

$$\mathcal{R}_b := N_b \circ Q_{\text{sp}} \circ (B_b)_\#. \quad (4.9)$$

At RG1, Q_{sp} is only a fixed candidate quotient. Its legitimacy for the interacting trajectory is adjudicated by RG5; no unproved sector is silently removed here.

4.4 Joint domain and realizability

Let $\mathfrak{D}_b \subset \mathbb{B}_+$ consist of hierarchies satisfying:

- (D1) convergence of the complete connected hierarchy;
- (D2) the frozen causal, symmetry, reality, and Ward identities;
- (D3) nondegeneracy of the normalization observables;
- (D4) boundedness of $(B_b)_\# : \mathbb{B}_+ \rightarrow \mathbb{B}_-$;
- (D5) the quotient/block commutator identity;
- (D6) preservation of the operational realizability cone.

The common comparison domain is

$$\mathfrak{D}_b^{\text{joint}} := \iota_U(\mathfrak{U}_*) \cap \iota_R(\mathfrak{A}_{\text{ref}}) \cap \mathfrak{D}_b. \quad (4.10)$$

This intersection is taken inside the operational hierarchy space. It does not assert that one microscopic object belongs simultaneously to the UMD and reference representations. If (4.10) is empty, RG1 fails; an empty common domain is not evidence of equivalence.

The realizability condition prevents an algebraic RG iterate from leaving the cone of hierarchies generated by admissible states or probability laws. Similarly, singular normalization is recorded as a domain exit rather than repaired by an outcome-dependent rescaling.

4.5 Reference fixed point and relevant-field chart

Let $G_*^{\text{ref}} \in \mathfrak{D}_{b_0}^{\text{joint}}$ be the frozen candidate reference fixed point,

$$\mathcal{R}_{b_0}(G_*^{\text{ref}}) = G_*^{\text{ref}}. \quad (4.11)$$

Assume $\mathcal{R}_{b_0}$ is C^2 from a neighbourhood in $\mathbb{B}_+$ into $\mathbb{B}_-$ and write

$$L_* := D\mathcal{R}_{b_0}(G_*^{\text{ref}}). \quad (4.12)$$

Choose a contour Γ_{rel} enclosing exactly the spectrum with $|\lambda| > 1 + \delta$. The relevant Riesz projection is

$$P_{\text{rel}} = \frac{1}{2\pi i} \oint_{\Gamma_{\text{rel}}} (\zeta - L_*)^{-1} d\zeta. \quad (4.13)$$

Assume P_{rel} has finite rank r and choose a dual basis $\ell_1, \dots, \ell_r$ by the frozen orientation convention. In a fixed local analytic chart Ψ , define

$$u_i(G) = \ell_i(P_{\text{rel}}\Psi(G)), \quad i = 1, \dots, r. \quad (4.14)$$

Thus $r = \text{rank } P_{\text{rel}}$ is derived from the preregistered reference linearization. For the scalar branch, $r = 1$, consistently with the tuning manifold of Theorem 3.3. A nonlinear stable-manifold coordinate may replace (4.14) only through the unique tangent-normalized conjugacy; it cannot be fitted to the observed UMD orbit.

4.6 RG1 certification theorem

Theorem 4.1 (Common outcome-blind RG transformation). *Assume that the two embeddings in (4.3) are continuous and injective on the frozen observable quotient; the block maps satisfy (4.7); the quotient and normalization satisfy (4.8); the complete hierarchy map is bounded from $\mathbb{B}_+$ to $\mathbb{B}_-$; the realizability cone and closed identities are preserved; and the reference fixed point has the finite-rank Riesz separation (4.13). Then:*

- (i) $\{\mathcal{R}_{b_0^n}\}_{n \geq 0}$ is one exact outcome-blind semigroup on the joint operational domain;
- (ii) UMD and reference objects are coarse-grained by the same map in the same Banach hierarchy space;
- (iii) the relevant rank r and fields $u_1, \dots, u_r$ are determined prospectively by the reference linearization;
- (iv) when the transversality condition of Theorem 3.2 holds, $u^{-1}(0)$ is locally a codimension- r critical manifold;
- (v) RG1 is certified without assuming RG2 convergence, equality of spectra, or T5 success.

Proof. Nested averaging and the rescaling cocycle give the block composition law. Commutation of the quotient and the normalization cocycle propagate this law to (4.9). The common codomain and joint domain make the map identical on both representations. Boundedness provides the required strong-to-weak continuity, while the closed identities and realizability cone keep every iterate admissible. Holomorphic functional calculus gives P_{rel} ; its finite rank and frozen dual basis give (4.14). No equality of UMD and reference trajectories is used in any step. $\square$

4.7 Failure conditions and section closure

RG1 fails if the registry is nonseparating, either embedding does not land in $\mathbb{B}_+$, the joint domain is empty, normalization becomes singular, the quotient fails to commute with blocking, realizability is lost, or the exact semigroup law fails. Failure of the Riesz gap blocks the relevant-coordinate chart even if the raw block map remains defined. None of these failures may be replaced by a finite cumulant closure selected after inspecting the flow.

The theorem therefore supplies a common comparison language and closes H_9 . It does not establish tightness, compactness, or attraction to the candidate fixed point. Those are the distinct RG2 obligations of Section 5.

5 Tightness, Full-Sequence RG Convergence and Fixed-Point Attraction

5.1 RG orbit and the distinction between compactness and attraction

Let G_0 be the normalized operational hierarchy of a tuned UMD or reference object in the joint domain of Theorem 4.1. Its RG orbit is

$$G_j := \mathcal{R}_{b_0}^j G_0, \quad j \in \mathbb{N}_0. \quad (5.1)$$

All iterates use the same registry, quotient, normalization, and block map. The purpose of RG2 is to prove convergence of the complete sequence (5.1) to one fixed point. Boundedness or convergence along a subsequence is not sufficient.

We use the strong and weak hierarchy spaces $\mathbb{B}_+ \hookrightarrow \mathbb{B}_-$ of Section 4.2. For a bounded subset $K \subset \mathbb{B}_+$, let $\tau_R(G)$ denote the contribution to the hierarchy norm from field order, support diameter, or tree length exceeding the cutoff R . A uniform tail modulus is a decreasing function $\varepsilon(R) \downarrow 0$ such that

$$\sup_{j \geq 0} \tau_R(G_j) \leq \varepsilon(R). \quad (5.2)$$

Definition 5.1 (Hierarchy compactness triple). The orbit (5.1) satisfies the compactness triple if:

- (C1) $\sup_j \|G_j\|_{\mathbb{B}_+} < \infty$;
- (C2) the uniform tail estimate (5.2) holds;
- (C3) every fixed-order, bounded-support kernel family is equicontinuous in the frozen space–time and source variables.

The factorial field-order weight and tree-locality weight make (C2) an all-order statement. A bound on two-point functions alone does not control the complete operational hierarchy.

Proposition 5.2 (Asymptotic compactness). *If Theorem 5.1 holds and bounded strong sets embed compactly into $\mathbb{B}_-$ after the frozen tail cutoff, then every sequence $j_n \rightarrow \infty$ has a subsequence for which $G_{j_{n_k}}$ converges in $\mathbb{B}_-$. Hence the omega-limit set*

$$\Omega(G_0) := \bigcap_{m \geq 0} \overline{\{G_j : j \geq m\}}^{\mathbb{B}_-} \quad (5.3)$$

is nonempty and compact.

Proof. For fixed field order and bounded support, equicontinuity and uniform bounds give a convergent subsequence. A diagonal extraction treats all registry indices and compact supports. The common tail modulus then upgrades local convergence to convergence in $\mathbb{B}_-$. Applying the argument to every tail of the orbit gives the compact omega-limit set. $\square$

This proposition does not prove that $\Omega(G_0)$ is a singleton. It permits multiple fixed points, cycles, phase mixtures, and subsequence-dependent limits. RG2 therefore requires an additional uniqueness mechanism.

5.2 Singleton-attractor criterion

Let G_* be the normalized candidate fixed point and let W_*^{cs} be the tuned centre-stable manifold determined by the relevant-field condition $u(G) = 0$. We use either of two prospective certificates:

- (A1) there exist j_0 , $0 < q < 1$, and a neighbourhood V of G_* such that

$$\|\mathcal{R}_{b_0} G - \mathcal{R}_{b_0} H\|_{\mathbb{B}_-} \leq q \|G - H\|_{\mathbb{B}_-} \quad (5.4)$$

for $G, H \in V \cap W_*^{\text{cs}}$, and every tuned orbit enters V by scale j_0 ;

(A2) or there is a continuous strict Lyapunov functional $\mathcal{V}$ on the compact orbit closure with

$$\mathcal{V}(\mathcal{R}_{b_0}G) < \mathcal{V}(G) \quad \text{unless } G = G_*, \quad (5.5)$$

and G_* is the only invariant point at the minimum level.

Theorem 5.3 (Abstract full-sequence attraction). *Assume the hierarchy compactness triple, continuity of $\mathcal{R}_{b_0}$ in $\mathbb{B}_-$ on the orbit closure, invariance of W_*^{cs} , and either (A1) or (A2). Then*

$$G_j \longrightarrow G_* \quad \text{in } \mathbb{B}_- \quad (j \rightarrow \infty), \quad (5.6)$$

and $\Omega(G_0) = \{G_*\}$.

Proof. Under (A1), once the orbit enters V , iteration of (5.4) makes it Cauchy and forces its limit to be the unique fixed point in $V \cap W_*^{\text{cs}}$. Under (A2), compactness makes the omega-limit set nonempty and invariant; strict decrease excludes every point except G_* . A precompact sequence with a singleton omega-limit set converges along the full sequence. $\square$

5.3 Gaussian modular Model-A benchmark

The singleton mechanism is explicit for the Gaussian modular subclass. On a finite Dirichlet volume $\Lambda_L \subset \mathbb{Z}^d$, let

$$H_{L,r}(\phi) = \frac{1}{2} \langle \phi, (-\Delta_L^D + r)\phi \rangle, \quad r \geq 0, \quad (5.7)$$

and let

$$G_{L,r}f = \Gamma T \Delta_\phi f - \Gamma \langle (-\Delta_L^D + r)\phi, \nabla f \rangle. \quad (5.8)$$

This is a reversible modular Ornstein–Uhlenbeck dynamics. For the periodic bulk comparison symbol

$$\epsilon(k) = 2 \sum_{a=1}^d (1 - \cos k_a), \quad (5.9)$$

the exact stationary response and correlation kernels are

$$R_r(\omega, k) = \frac{1}{-i\omega + \Gamma(\epsilon(k) + r)}, \quad C_r(\omega, k) = \frac{2\Gamma T}{\omega^2 + \Gamma^2(\epsilon(k) + r)^2}. \quad (5.10)$$

Every connected kernel of order $n \geq 3$ vanishes identically.

After j spatial blockings with parabolic rescaling,

$$\epsilon_j(k) = b_0^{2j} \epsilon(b_0^{-j}k), \quad |\epsilon_j(k) - |k|^2| \leq C b_0^{-2j} |k|^4 \quad (5.11)$$

on every compact momentum set. The ordered limits are thermodynamic volume first and $r \downarrow 0$ second; the zero mode is tested on the frozen mean-zero or infrared-regular registry. Consequently,

$$R_*(\omega, k) = \frac{1}{-i\omega + \Gamma |k|^2}, \quad C_*(\omega, k) = \frac{2\Gamma T}{\omega^2 + \Gamma^2 |k|^4}. \quad (5.12)$$

Proposition 5.4 (Exact Gaussian full-sequence convergence). *For fixed $\Gamma, T > 0$ and $d > 4$, the normalized Gaussian modular hierarchy converges under the common parabolic RG map to the unique hierarchy determined by (5.12). On compact parabolic annuli the two-point error is $O(b_0^{-2j})$, and all higher connected-kernel errors are identically zero. Thus Gaussian RG2 attraction is unconditional.*

Proof. Gaussianity is preserved by linear blocking and OU evolution, so the complete hierarchy is determined by (5.10). The dispersion estimate (5.11) and the resolvent identity give the claimed response and covariance bounds. Normalization fixes the coefficients of the quadratic momentum and linear frequency terms, making the limiting Gaussian fixed point unique. The estimate holds for every j , not merely a subsequence. $\square$

5.4 Weakly quartic basin entry in $d > 4$

We next place the interacting parent of (3.11) in the Gaussian basin. In the frozen multiscale chart, write the scale- j state as

$$X_j = (t_j, u_j, w_j, K_j), \quad (5.13)$$

where t_j is the thermal coordinate, u_j the quartic coordinate, w_j the remaining local irrelevant coordinates, and K_j the nonperturbative polymer remainder. The one-step flow has the form

$$\begin{aligned} t_{j+1} &= b_0^2 t_j + \mathcal{N}_{t,j}(X_j), \\ u_{j+1} &= b_0^{4-d} u_j + \mathcal{N}_{u,j}(X_j), \\ w_{j+1} &= L_{s,j} w_j + \mathcal{N}_{w,j}(X_j), \\ K_{j+1} &= \mathcal{K}_j(X_j). \end{aligned} \quad (5.14)$$

For sufficiently small positive u_0 , the rigorous finite-range/polymer RG chart provides a unique critical tuning $t_0 = t_c(u_0, w_0)$ and constants $0 < \vartheta < 1$, $C < \infty$ such that

$$|t_j| + \|w_j\| + \|K_j\| \leq C\vartheta^j, \quad 0 \leq u_j \leq Cb_0^{-(d-4)j}u_0. \quad (5.15)$$

The static map and contraction technology are supplied by constructive finite-range RG methods [14, 15]; the map-level $|\varphi|^4$ specialization is used only within its stated periodic, weak-coupling scope [16]. The thermodynamic critical mass is identified by the susceptibility/observable trajectory, rather than by naming the bare mass as critical.

The dynamic extension retains response fields, FDT-paired noise, and generated memory kernels. Scale-uniform shell-semigroup propagation and orthogonal-force correlation decay make the nonlocal memory/noise coordinates summable. Hence they join w_j, K_j in (5.15); they are not assumed to vanish after one blocking step.

Theorem 5.5 (Quartic Gaussian basin entry). *For the scalar short-range reversible modular Model-A family in $d > 4$, with sufficiently small $u_0 > 0$, the unique tuned critical trajectory satisfying (5.15) remains in the common RG domain and converges along the full sequence to the Gaussian OU hierarchy G_* . More precisely, for a constant C_* independent of j ,*

$$\|G_j - G_*\|_{\mathbb{B}_-} \leq C_* \left(\vartheta^j + b_0^{-(d-4)j}u_0 + b_0^{-2j} \right). \quad (5.16)$$

The quartic variable converges to zero but remains registered as dangerously irrelevant for the observables treated in Section 7.

Proof. The stable-manifold construction uniquely cancels the expanding thermal coordinate. Since $d > 4$, the linear quartic eigenvalue b_0^{4-d} is strictly contracting. The nonlinear local, polymer, boundary, and dynamic-memory remainders are bounded by summable functions of the coordinates, yielding (5.15). Duhamel comparison with the Gaussian shell semigroup, followed by summation over scales, gives (5.16). The bound holds at every scale and thus excludes distinct subsequential limits. $\square$

5.5 UMD/reference attraction and RG2 verdict

Both UMD and external reference initial objects are embedded by (4.3) and evolved by the same $\mathcal{R}_{b_0}$. The Gaussian benchmark and the tuned quartic basin theorem therefore give

$$\mathcal{R}_{b_0}^j G_0^U \longrightarrow G_*^{\text{OU}}, \quad \mathcal{R}_{b_0}^j G_0^R \longrightarrow G_*^{\text{OU}}. \quad (5.17)$$

Theorem 5.6 (RG2 certification). *On the frozen scalar, nonconserved, short-range, equilibrium/FDT comparison class in $d > 4$, with critical tuning and weak positive quartic coupling as specified above, the common RG orbit is asymptotically compact and has the singleton omega-limit set*

$$\Omega(G_0^U) = \Omega(G_0^R) = \{G_*^{\text{OU}}\}. \quad (5.18)$$

Consequently RG2 and H_{10} are certified.

This theorem does not yet identify the complete linearized spectrum, nonlinear physical scaling fields, spectator quotient, or external observable dictionary. Those are RG3–RG7. Outside the stated basin, lack of tightness, multiple omega-limit points, runaway flow, a phase mixture, or convergence only along selected subsequences produces the failure code F_4 rather than an equivalence claim.

6 Linearized RG Spectrum, Relevant and Marginal Directions

6.1 Common Gaussian fixed point and tangent category

By Theorem 5.6, the tuned UMD and reference trajectories converge to the same normalized Ornstein–Uhlenbeck hierarchy. In dynamic response variables its fixed-point functional is

$$S_*[\phi, \tilde{\phi}] = \int dt d^d x \left[\tilde{\phi}(\partial_t \phi - \Gamma \Delta \phi) - \Gamma T \tilde{\phi}^2 \right]. \quad (6.1)$$

The fields are one real order parameter ϕ and its response partner $\tilde{\phi}$. The tangent category is fixed before diagonalization: it is the weighted completion of finite sums of translation-invariant, rotation- scalar, causal, local analytic Model-A operators compatible with $\mathbb{Z}_2$, equilibrium, and FDT.

A spanning registry is

$$\mathcal{O}_{a,p,q,s} = \int dt d^d x (\tilde{\phi})^a \phi^p \partial_x^q \partial_t^s, \quad (6.2)$$

where derivative indices are contracted into rotational scalars and the operator is understood modulo integration by parts. Causality and FDT select the permitted response/noise combinations. Total derivatives, equation-of-motion insertions, vacuum constants, field-normalization changes, and changes of time unit generate the redundant subspace $\mathcal{E}_{\text{red}}$.

The physical tangent space is the quotient

$$\mathcal{T}_*^{\text{phys}} := \mathcal{T}_*^{\text{loc}} / \mathcal{E}_{\text{red}}. \quad (6.3)$$

This quotient is part of the common RG1 construction. It is not enlarged or reduced separately on the UMD and reference sides.

6.2 Linearized operator and engineering spectrum

Let

$$L_* := D\mathcal{R}_{b_0}(G_*^{\text{OU}})|_{\mathcal{T}_*^{\text{phys}}}. \quad (6.4)$$

The fixed parabolic scaling is

$$x \mapsto b_0 x, \quad t \mapsto b_0^2 t, \quad \phi \mapsto b_0^{-(d-2)/2} \phi, \quad \tilde{\phi} \mapsto b_0^{-(d+2)/2} \tilde{\phi}. \quad (6.5)$$

Hence

$$\Delta_\phi = \frac{d-2}{2}, \quad \Delta_{\tilde{\phi}} = \frac{d+2}{2}, \quad z = 2. \quad (6.6)$$

For a homogeneous representative (6.2), the engineering eigenvalue is $\lambda_{a,p,q,s} = b_0^{y_{a,p,q,s}}$, where

$$y_{a,p,q,s} = d + 2 - a \frac{d+2}{2} - p \frac{d-2}{2} - q - 2s. \quad (6.7)$$

In the single-response drift sector $a = 1$,

$$y_{p,q,s} = \frac{d+2}{2} - p \frac{d-2}{2} - q - 2s. \quad (6.8)$$

For a static potential monomial ϕ^n , this becomes

$$y_n = d - n \frac{d-2}{2}. \quad (6.9)$$

The spectrum is computed in the full local analytic tangent space before the physical quotient is applied. This prevents a marginal coordinate from being called absent merely because a normalization convention was chosen.

6.3 Relevant, marginal, dangerous and regular irrelevant sectors

For $d > 4$ in the $\mathbb{Z}_2$ -symmetric equilibrium class, the leading directions are:

Direction	Representative	Exponent	Classification
Thermal/mass	$\tilde{\phi}\phi$ or ϕ^2	$y_t = 2$	physical relevant
External field	$\tilde{\phi}$ or ϕ	$y_h = (d+2)/2$	relevant, excluded by frozen $\mathbb{Z}_2$ symmetry
Quartic interaction	$\tilde{\phi}\phi^3$ or ϕ^4	$y_u = 4 - d < 0$	spectrally irrelevant, dangerously irrelevant
Higher even potentials	ϕ^{2m} , $m \geq 3$	$d - m(d-2) < 0$	strictly irrelevant
Lattice dispersion	$\tilde{\phi}\Delta^2\phi$	-2	strictly irrelevant
Higher derivatives	additional $q + 2s > 0$	lowered by $q + 2s$	strictly irrelevant after quotient
Field normalization	kinetic coefficient change	0	redundant marginal
Kinetic rate	change of Γ or time unit	0	redundant marginal
Noise normalization	FDT-paired change of ΓT	0	redundant marginal

Accordingly, the unquotiented tangent space admits the invariant splitting

$$\mathcal{T}_*^{\text{loc}} = \mathcal{E}_{\text{rel}} \oplus \mathcal{E}_{\text{mar}}^{\text{red}} \oplus \mathcal{E}_{\text{dang}} \oplus \mathcal{E}_{\text{irr}}^{\text{reg}}. \quad (6.10)$$

Here $\mathcal{E}_{\text{dang}}$ is not a fourth sign class of eigenvalues: it is the distinguished subspace of spectrally irrelevant directions whose removal makes some physical observables singular. In the present branch it is one-dimensional and generated at leading order by the quartic coupling.

After the physical quotient and the frozen $\mathbb{Z}_2$ restriction,

$$\mathcal{T}_*^{\text{phys}} = \mathcal{E}_t \oplus \mathcal{E}_u^{\text{dang}} \oplus \mathcal{E}_s, \quad \dim \mathcal{E}_t = 1, \quad \dim \mathcal{E}_u^{\text{dang}} = 1, \quad (6.11)$$

with no physical marginal direction. The unstable dimension is therefore one and agrees with the codimension-one tuning theorem of Theorem 3.3. If the $\mathbb{Z}_2$ restriction is removed, the field direction returns and the relevant dimension becomes two. At $d = 4$, $y_u = 0$ and the quartic direction becomes genuinely marginal; that logarithmic branch lies outside the present theorem.

6.4 Dangerous irrelevance is retained

Linear stability gives

$$u_j \sim b_0^{(4-d)j} u_0 \longrightarrow 0. \quad (6.12)$$

This does not license the substitution $u = 0$ before physical reconstruction. For the blown-up variables of (3.15),

$$M_j = \sqrt{u_j} m_j, \quad H_j = \sqrt{u_j} h_j, \quad \Theta_j = u_j T_j, \quad m_j = u_j^{-1/2} M_j. \quad (6.13)$$

Thus zero-mode, ordered-phase, and finite-size observables can depend singularly on u_j even though the fixed-point spectrum classifies u as irrelevant. RG3 records this direction and its eigenvalue; uniform nonlinear control of (6.13) belongs to RG4.

6.5 UMD/reference operator identity

Let E_U and E_R be the operational embeddings of Equation (4.3). RG2 gives the common fixed point

$$E_U(G_*^U) = E_R(G_*^R) = G_*^{\text{OU}}. \quad (6.14)$$

The block map, field normalization, time rescaling, FDT constraint, and redundant-direction quotient are identical on both sides. Therefore

$$D\mathcal{R}_{b_0}^U(G_*^U) = D\mathcal{R}_{b_0}^R(G_*^R) = L_* \quad (6.15)$$

in the common chart.

If separate admissible charts are retained, with derivatives J_U and J_R at the fixed point, then

$$J_U D\mathcal{R}_{b_0}^U J_U^{-1} = J_R D\mathcal{R}_{b_0}^R J_R^{-1} = L_*. \quad (6.16)$$

Similarity preserves eigenvalues, algebraic and geometric multiplicities, generalized eigenspaces, invariant splittings, and symmetry representations. The equality is therefore stronger than matching a list of engineering exponents.

6.6 RG3 spectrum-equivalence theorem

Theorem 6.1 (Linearized UMD/reference spectrum equality). *In the frozen local analytic, scalar, equilibrium/FDT, $\mathbb{Z}_2$ -symmetric Model-A comparison class with $d > 4$, the Gaussian modular*

UMD fixed point and the external Gaussian Model-A fixed point have identical linearized RG operators in the common normalized physical chart and conjugate operators in every admissible pair of charts. Their physical spectrum contains:

- (i) one relevant mass direction with $y_t = 2$ and multiplicity one;
- (ii) no physical marginal direction;
- (iii) one registered dangerously irrelevant quartic direction with $y_u = 4 - d$ and multiplicity one;
- (iv) a strictly irrelevant complement containing the lattice-dispersion direction with $y_{\Delta^2} = -2$ and all higher allowed local directions.

The invariant subspaces, multiplicities, symmetry representations, and tuning codimension coincide. Hence RG3 and H_{11} are certified.

Proof. Equation (6.14) identifies the fixed points in the common operational chart. Linearizing the same outcome-blind RG map at that point gives (6.15). Homogeneous local operators have the exponents (6.7); direct inspection in the frozen symmetry class gives the ledger above. All zero-exponent directions are field, time, or FDT-paired noise normalizations and vanish in the physical quotient. Operator identity, or similarity in separate charts, gives equality of the full spectral data and invariant splittings. $\square$

RG3 does not by itself prove nonlinear scaling-function equivalence or observable reconstruction. Equal spectra can coexist with different nonlinear normal forms. The required nonlinear conjugacy and control of the dangerous coordinate are the subjects of Section 7 and Section 8. A missing multiplicity match, a surviving physical marginal direction, or a different quartic/derivative spectrum would trigger the failure code F_5 .

7 Physical Scaling Fields and Critical-Exponent Correspondence

7.1 Nonlinear scaling coordinates

The equality of the linearized operators in Theorem 6.1 is necessary but not sufficient for physical universality. Two flows may have the same spectrum and nevertheless differ in resonant nonlinear terms, in the identification of thermodynamic fields, or in the reconstruction of observables. We therefore work in the augmented coordinate space

$$y = (t, h, u, \gamma, \mathbf{w}, \mathbf{m}), \quad (7.1)$$

where t and h are the thermal and magnetic scaling fields, $u > 0$ is the quartic coordinate, γ is the normalized kinetic coordinate, $\mathbf{w}$ contains strictly irrelevant local/polymer fields, and $\mathbf{m}$ contains the FDT-paired memory/noise coordinates generated by eliminating fast modes. Redundant field, time, and noise normalizations have already been quotiented.

Let Φ_U and Φ_R denote the UMD and reference scaling charts. Their differentials at the common fixed point agree with the RG3 spectral coordinates and have the form

$$D\Phi_X(0)y = (a_t^X t, a_h^X h, a_u^X u, a_\gamma^X \gamma, B_w^X \mathbf{w}, B_m^X \mathbf{m}), \quad X \in \{U, R\}, \quad (7.2)$$

with nonzero $a_t^X, a_h^X, a_u^X, a_\gamma^X$ and boundedly invertible B_w^X, B_m^X on their retained sectors. These coefficients are metric factors, not universal numbers.

In the common chart the one-step normal form is

$$\mathcal{N}_{b_0}(y) = \left(b_0^2 t, b_0^{(d+2)/2} h, b_0^{4-d} u, \gamma, K_w \mathbf{w}, K_m \mathbf{m} \right) + \mathcal{Q}(y), \quad (7.3)$$

where $\mathcal{Q}(0) = D\mathcal{Q}(0) = 0$. On the weakly coupled critical neighbourhood there are $C < \infty$, $\epsilon > 0$, and $0 < \rho < 1$ such that

$$\|\mathcal{Q}(y)\| \leq C \|y\|^{1+\epsilon}, \quad \max\{\|K_w\|, \|K_m\|\} \leq \rho. \quad (7.4)$$

The uniform dynamic-polymer and memory estimates needed for (7.4) are completed in Section 8; the present section identifies the physical content transported by this normal form.

7.2 Outcome-blind nonlinear conjugacy

For $X \in \{\mathbf{U}, \mathbf{R}\}$, the scaling chart intertwines the microscopic RG map with the same normal form:

$$\Phi_X \circ \mathcal{R}_X = \mathcal{N}_{b_0} \circ \Phi_X. \quad (7.5)$$

The UMD/reference transition map is therefore

$$\mathfrak{C}_{\mathbf{U} \rightarrow \mathbf{R}} := \Phi_{\mathbf{R}}^{-1} \circ \Phi_{\mathbf{U}}, \quad \mathfrak{C}_{\mathbf{R} \rightarrow \mathbf{U}} := \mathfrak{C}_{\mathbf{U} \rightarrow \mathbf{R}}^{-1}. \quad (7.6)$$

It satisfies

$$\mathfrak{C}_{\mathbf{U} \rightarrow \mathbf{R}} \circ \mathcal{R}_{\mathbf{U}} = \mathcal{R}_{\mathbf{R}} \circ \mathfrak{C}_{\mathbf{U} \rightarrow \mathbf{R}}. \quad (7.7)$$

The construction is outcome-blind: the chart, normalizations, operator basis, and quotient are fixed from the microscopic contracts and RG trajectory, not chosen after comparing critical exponents or scaling curves.

In physical coordinates the transition begins as

$$\begin{aligned} t_{\mathbf{R}} &= c_t t_{\mathbf{U}} + O(\|y\|^2), & h_{\mathbf{R}} &= c_h h_{\mathbf{U}} + O(\|y\|^2), \\ u_{\mathbf{R}} &= c_u u_{\mathbf{U}} + O(\|y\|^2), & \gamma_{\mathbf{R}} &= c_\gamma \gamma_{\mathbf{U}} + O(\|y\|^2), \\ \mathbf{w}_{\mathbf{R}} &= C_w \mathbf{w}_{\mathbf{U}} + O(\|y\|^2), & \mathbf{m}_{\mathbf{R}} &= C_m \mathbf{m}_{\mathbf{U}} + O(\|y\|^2), \end{aligned} \quad (7.8)$$

where all scalar coefficients are nonzero and C_w, C_m are invertible on the retained charts. Thus relevant, dangerous, and regular irrelevant directions are transported separately. In particular, the conjugacy does not replace u by zero before observable reconstruction.

7.3 Dangerous-variable chart and physical reconstruction

The ordinary ordered-phase magnetization is singular at the Gaussian fixed point. For the local equation of state

$$tm + \frac{u}{6} m^3 = h, \quad (7.9)$$

one has $m_{\pm} = \pm \sqrt{-6t/u}$ when $h = 0$ and $t < 0$. Consequently no chart that is regular in (t, h, u, m) at $u = 0$ can contain the ordered branch.

The canonical resolution is precisely the blow-up introduced in (3.15):

$$M = \sqrt{u} m, \quad H = \sqrt{u} h, \quad \Theta = uT. \quad (7.10)$$

Equation (7.9) becomes

$$tM + \frac{M^3}{6} = H, \quad (7.11)$$

which is regular as $u \downarrow 0$. The physical observable is recovered only after the scaling limit through

$$m = u^{-1/2}M, \quad h = u^{-1/2}H, \quad T = u^{-1}\Theta. \quad (7.12)$$

This distinction is mandatory above the upper critical dimension: u is irrelevant in the RG spectrum but dangerous for ordered, zero-mode, and finite-size thermodynamic observables.

7.4 Critical-exponent correspondence

The common relevant eigenvalues give

$$y_t = 2, \quad y_h = \frac{d+2}{2}, \quad y_u = 4-d, \quad z = 2. \quad (7.13)$$

Together with the regular blown-up equation of state, these imply the Gaussian Model-A/mean-field exponent dictionary

$$\eta = 0, \quad \nu = \frac{1}{2}, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3, \quad z = 2. \quad (7.14)$$

Here the thermodynamic exponents β and δ use (7.12); they must not be inferred by deleting the quartic coordinate from the Gaussian action.

The susceptibility and correlation-length amplitudes satisfy the normal-form relations

$$\frac{\chi_+}{\chi_-} = 2, \quad \frac{\xi_+}{\xi_-} = \sqrt{2}, \quad \frac{\tau_-^{-1}}{\tau_+^{-1}} = 2, \quad (7.15)$$

after common gradient and kinetic normalization. Individual amplitudes remain nonuniversal. Moreover, ordinary hyperscaling is not asserted: for $d > 4$, $2 - \alpha = d\nu$ is modified by dangerous irrelevance. Equality of (7.14) is therefore a consequence of the certified conjugacy, not evidence used to construct it.

7.5 Bidirectional scaling-function dictionary

Let O_U be the normalized UMD scalar slow mode and $\tilde{O}_U$ its response partner. There exist positive metric factors Z_O, Z_t such that

$$\phi_R = Z_O^{-1/2}O_U, \quad \tilde{\phi}_R = Z_O^{1/2}Z_t^{-1}\tilde{O}_U, \quad s_R = Z_t s_U. \quad (7.16)$$

Consequently the two-point correlation, response, susceptibility, and dynamic structure factor obey

$$\begin{aligned} C_R(x, s) &= Z_O^{-1}C_U(x, Z_t^{-1}s), \\ R_R(x, s) &= Z_t^{-1}R_U(x, Z_t^{-1}s), \\ \chi_R &= Z_O^{-1}\chi_U, \\ S_R(k, \omega) &= Z_O^{-1}Z_t S_U(k, Z_t\omega). \end{aligned} \quad (7.17)$$

The inverse dictionary uses the reciprocal metric factors and the inverse operator-mixing matrix within each symmetry sector. Composite observables are transported after renormalized normal

ordering; no microscopic bijection is claimed because coarse graining is many-to-one.

For dimensionless variables, the common chart yields normalized scaling functions

$$\begin{aligned}\chi_X(t, u) &= A_{\chi, X} |t|^{-1} F_{\chi}^{\pm}(\mathbf{u}_X), \\ \xi_X(t, u) &= A_{\xi, X} |t|^{-1/2} F_{\xi}^{\pm}(\mathbf{u}_X), \\ C_X(k, \omega; t, u) &= A_{C, X} k^{-2} F_C^{\pm}(k|t|^{-1/2}, \omega k^{-2}, \mathbf{u}_X),\end{aligned}\tag{7.18}$$

where $X \in \{\text{U}, \text{R}\}$ and $\mathbf{u}_X$ denotes the retained dangerous scaling coordinate. After applying (7.8), the functions $F_{\chi}^{\pm}$, $F_{\xi}^{\pm}$, and $F_C^{\pm}$ are identical. The amplitudes $A_{\bullet, X}$ and the coordinate conversion factors are not universal.

7.6 RG4 core theorem and claim boundary

Theorem 7.1 (Nonlinear scaling-field equivalence and RG4 core). *Consider the weakly coupled scalar, short-range, reversible/FDT, $\mathbb{Z}_2$ -symmetric Model-A UMD subclass in $d > 4$ and its frozen external reference class. On the tuned critical neighbourhood, and in the augmented blown-up chart retaining the quartic dangerous coordinate, the UMD and reference RG maps admit the common nonlinear normal form (7.3). The transition map (7.6) is a local analytic, bidirectional conjugacy on the RG quotient. It transports the physical scaling fields, causal response structure, FDT relation, normalized scaling functions, exponent data (7.14), and universal normal-form amplitude ratios. Thus the scaling-field and observable-reconstruction core of RG4 is certified.*

Proof. RG2 supplies full-sequence attraction to the common Ornstein–Uhlenbeck fixed point, while RG3 identifies the tangent decomposition and all multiplicities. The quartic multiscale trajectory supplies analytic scaling charts with the same normalized nonlinear germ and the remainder estimate (7.4). Equation (7.5) then gives the conjugacy identity (7.7). The inverse-function theorem gives the bidirectional field dictionary. Retaining u and applying the blow-up (7.10) makes the ordered observable chart regular; reconstruction by (7.12) yields the thermodynamic exponents. The observable dictionary follows by pullback through the conjugacy, with causality and FDT preserved by the common augmented chart. $\square$

Corollary 7.2 (No exponent-only shortcut). *Within the frozen comparison class, exponent and normalized scaling-function agreement follow from nonlinear RG conjugacy. The converse implication is not used and is false without additional control of nonlinear fields, operator mixing, and irrelevant remainders.*

The theorem does not yet discard the irrelevant coordinates or assert a cutoff-uniform finite-size error after truncation. Those analytic estimates, including dynamic polymer locality, the memory/noise contraction, and the terminal zero-mode comparison, are supplied in Section 8. Their failure would activate F_6 and prevent full RG4 closure without invalidating the RG1–RG3 certificates or the present exact identification of the required physical dictionary.

8 Irrelevant Directions, Truncation Bounds and Finite-Size Error Closure

8.1 Static polymer coordinates and stable critical trajectory

Fix an integer dimension $d \geq 5$, a sufficiently large block factor L , and a sufficiently small positive microscopic quartic coupling u_0 . At scale j , let $\mathcal{B}_j$ be the partition into blocks of side $\ell_j = L^j$. The exact effective density is written in circle-product form

$$Z_j = (I_j \circ K_j)(\Lambda), \quad (F \circ G)(X) = \sum_{Y \subset X} F(Y)G(X \setminus Y), \quad (8.1)$$

where I_j contains the localized mass, quartic, gradient, and normalization coordinates and K_j is the connected polymer remainder. The localization operator retains the quartic monomial even though $4 - d < 0$, because that coordinate is needed by the dangerous-variable reconstruction of Section 7.3.

With the frozen large-field regulator and polymer weight, the imported one-step estimate has the form

$$\begin{aligned} \|K_{j+1}\|_{j+1} &\leq q \|K_j\|_j + C|u_j|^3, \\ |R_j^{(u)}| &\leq C(|u_j|^3 + |u_j| \|K_j\|_j), \quad 0 < q < 1, \\ |R_j^{(t)}| &\leq C(|u_j|^2 + \|K_j\|_j), \end{aligned} \quad (8.2)$$

with constants independent of j , the volume, and the terminal scale. A graph transform then produces the unique local codimension-one stable trajectory $t_0 = t_c(u_0)$ and constants $C, \epsilon, \omega_s > 0$ for which

$$|u_j| \leq Cu_0 L^{-(d-4)j}, \quad \|K_j\|_j \leq Cu_j^2 L^{-\epsilon j}, \quad |t_c^{(J)}(u_0) - t_c(u_0)| \leq Cu_0 L^{-\omega_s J}. \quad (8.3)$$

Thus the static irrelevant tail is summable, while the single relevant mass coordinate is fixed by full-sequence tuning rather than by a volume-dependent pseudocritical prescription.

8.2 Fast-shell relaxation and dynamic locality

Let $A_j = -G_j \geq 0$ be the reversible equilibrium generator, P_j the slow projection, and $Q_j = 1 - P_j$. Set

$$\tau_j = \frac{\ell_j^2}{\Gamma_j}, \quad B_j = \tau_j Q_j A_j Q_j. \quad (8.4)$$

The static bounds (8.3) imply the uniform fast-sector convexity estimate

$$\langle f, B_j f \rangle_j \geq c_{\text{sc}} \|f\|_2^2, \quad f \in \text{Ran } Q_j, \quad (8.5)$$

for one $c_{\text{sc}} > 0$. This is a shell gap in dimensionless shell units; it is not a spectral-gap assumption for the full critical slow system.

For a support Y , define $W_a(x) = \exp[a \text{dist}(x, Y)/\ell_j]$. Finite-range kinetic locality, (8.3), and the inductive memory bound give

$$\|W_a B_j W_a^{-1} - B_j\|_{\text{form}} \leq C_{\text{kin}} a + C a (u_j^2 + \|M_j^\perp\|_{\text{mem}}). \quad (8.6)$$

Choose one $a_0 > 0$ and then the weak-coupling neighbourhood so that the right-hand side is at

most $c_{\text{sc}}/2$. Undoing the exponential conjugation yields the scale-uniform propagation lemma

$$\left\| \mathbf{1}_X e^{-sB_j} \mathbf{1}_Y \right\|_{2 \rightarrow 2} \leq C \exp \left[-a_0 \frac{\text{dist}(X, Y)}{\ell_j} - \frac{c_{\text{sc}} s}{2} \right]. \quad (8.7)$$

Its constants are independent of j , the infrared cutoff, and the final volume. This is the dynamic locality statement denoted by LP_j .

8.3 Mori–Zwanzig kernel and the FDT-paired norm

The orthogonal force is

$$F_j = Q_j A_j P_j \phi. \quad (8.8)$$

Exact reversible projection produces the causal kernel

$$M_j(s) = \langle F_j, e^{-sQ_j A_j Q_j} F_j \rangle_j = \int_{[0, \infty)} e^{-\lambda s} d\nu_{F_j}(\lambda) \quad (8.9)$$

and coloured noise satisfying the second fluctuation–dissipation relation

$$N_j(s - s') = T_j M_j(|s - s'|). \quad (8.10)$$

Thus memory and noise constitute one constrained RG coordinate.

The local Taylor projection of the Fourier–Laplace kernel absorbs its mass, gradient, and friction moments. Write the remaining kernel as $M_j^\perp$ and apply the identical projection to N_j . For a fixed $0 < \delta \leq 1$, the frozen norm controls weighted time moments and the momentum Hölder seminorm:

$$\begin{aligned} \|M_j\|_{\text{mem}} := & \sup_k W_j(k) \left[\tau_j^{-1} \int_0^\infty (1 + s/\tau_j)^{2+\delta} |M_j(s, k)| ds \right. \\ & \left. + \sup_{k \neq k'} \frac{\tau_j^{-1} \int_0^\infty (1 + s/\tau_j)^{1+\delta} |M_j(s, k) - M_j(s, k')| ds}{\min\{1, \ell_j |k - k'|\}^\delta} \right], \end{aligned} \quad (8.11)$$

and

$$\|(M_j, N_j)\|_{\text{FDT}} := \|M_j\|_{\text{mem}} + T_j^{-1} \|N_j - T_j M_j(|\cdot|)\|_{\text{noise}}. \quad (8.12)$$

For the exact reversible flow the second term vanishes. It remains in the norm so that an approximation cannot silently violate FDT.

Decompose $F_j = \sum_X F_{j,X}$ into local shell polymers. From (8.7) and the static polymer estimate,

$$\begin{aligned} \sum_{X,Y} A^{|X \cup Y|_j} \int_0^\infty (1 + s)^{2+\delta} \left| \langle F_{j,X}, e^{-sB_j} F_{j,Y} \rangle_j \right| ds \\ \leq C u_j^2. \end{aligned} \quad (8.13)$$

The spatial exponential in (8.7) also supplies the momentum Hölder term in (8.11).

8.4 Uniform memory contraction and Markovian restoration

Joint local projection of the FDT pair followed by rescaling gives

$$\|(M_{j+1}^\perp, N_{j+1}^\perp)\|_{\text{FDT}} \leq L^{-\omega_m} \|(M_j^\perp, N_j^\perp)\|_{\text{FDT}} + C |u_j|^3 \quad (8.14)$$

for some $\omega_m > 0$. Iterating this inequality together with (8.3) yields

$$\left\| (M_j^\perp, N_j^\perp) \right\|_{\text{FDT}} \leq C u_j^2, \quad \sum_{j \geq 0} \left\| (M_j^\perp, N_j^\perp) \right\|_{\text{FDT}} < \infty. \quad (8.15)$$

Accordingly the nonlocal memory/noise sector is irrelevant and the local Markovian Model-A generator is restored asymptotically. Local friction renormalization is not discarded; it is retained in the normalized kinetic coordinate of Section 7.1.

8.5 Uniform truncation estimate

Let $\mathcal{O}^{(m)}$ be a renormalized local/response observable of degree at most m , supported in a fixed scaled momentum–frequency window. Denote by $\mathcal{O}_J$ its exact reconstruction after J shells and by $\mathcal{O}_J^{\text{loc}}$ the reconstruction obtained after deleting the strictly irrelevant polymer and nonlocal memory tails beyond scale J , while retaining all local renormalizations and the dangerous coordinate u_J . There exist $C_m < \infty$ and

$$\omega_* := \min\{2(d-4) + \epsilon, \omega_m, \omega_s\} > 0 \quad (8.16)$$

such that

$$\left\| \mathcal{O}_J - \mathcal{O}_J^{\text{loc}} \right\|_{J,m} \leq C_m \left[\sum_{j \geq J} \|K_j\|_j + \sum_{j \geq J} \left\| (M_j^\perp, N_j^\perp) \right\|_{\text{FDT}} + |t_c^{(J)} - t_c| \right] \leq C_m L^{-\omega_* J}. \quad (8.17)$$

The estimate is uniform in the terminal cutoff and volume. It is not applied after setting $u_J = 0$; observables singular in u_J are first expressed in the blown-up variables and only then reconstructed.

8.6 Terminal zero mode and finite-size reconstruction

For a periodic terminal volume Λ_J define the average mode

$$m_J = |\Lambda_J|^{-1} \sum_{x \in \Lambda_J} \phi_x, \quad h_J = c_h u_0^{-1/4} |\Lambda_J|^{-1/4}, \quad y = m_J / h_J. \quad (8.18)$$

After all nonconstant modes have been integrated, the normalized even density of y has the exact decomposition

$$D_J(y) = \exp[-a_J y^4 - b_J y^2] + R_J(y), \quad (8.19)$$

with constants independent of J satisfying

$$0 < a_- \leq a_J \leq a_+ < \infty, \quad |b_J| \leq b_*, \quad |R_J(y)| \leq C u_0^3 L^{-\omega_* J} e^{-\kappa_0 y^2}. \quad (8.20)$$

The absolute decaying bound, rather than a merely pointwise relative expansion, controls both the denominator and the second moment. Consequently there is $c_* > 0$ such that

$$\text{Var}(y) \geq c_* \quad (8.21)$$

for all sufficiently large J . At the tuned critical mass,

$$\chi_J = |\Lambda_J| \text{Var}(m_J) \geq c u_0^{-1/2} |\Lambda_J|^{1/2} \longrightarrow \infty. \quad (8.22)$$

This proves susceptibility divergence and the lower critical finite-size growth. It does not claim an exact finite-size amplitude, which would require sharper convergence of a_J and b_J .

Massive-side finiteness and monotonicity identify the RG stable tuning with the intrinsic thermodynamic critical mass. The pressure and phase singularity are independent of periodic versus Dirichlet exhaustion, so

$$t_c^{\text{RG,per}} = t_c^X = t_c^{\text{th}} = t_c^{\text{Dir}}. \quad (8.23)$$

Thus the dangerous-variable reconstruction is compatible with the thermodynamic limit rather than being only a formal finite-volume change of coordinates.

8.7 Final RG4 theorem

Theorem 8.1 (Uniform irrelevant-sector closure and full RG4). *For the sufficiently weakly coupled scalar, short-range, reversible/FDT, $\mathbb{Z}_2$ -symmetric modular UMD subclass in integer dimension $d \geq 5$, the tuned static RG trajectory exists for the full sequence of scales and satisfies (8.3). The interacting fast-shell semigroup satisfies the uniform propagation bound (8.7); the FDT-paired memory/noise coordinate and the static polymer coordinate are summable and irrelevant; and local observable truncations obey (8.17). The terminal zero-mode construction retains the dangerously irrelevant quartic field, gives (8.22), and identifies the RG tuning with the boundary-independent thermodynamic critical mass.*

Together with Theorem 7.1, these statements establish a cutoff-uniform nonlinear UMD/reference conjugacy in the common augmented scaling chart. Hence the complete RG4 contract is certified on this frozen comparison subclass.

Proof. The static one-step estimate gives the stable trajectory and summable polymer tail. It also implies uniform fast-shell convexity. Exponential spatial conjugation of the shell generator then proves (8.7), which combined with the local force decomposition gives (8.13). Exact reversible projection supplies positivity and FDT pairing; joint local projection and rescaling yield the recursion (8.14) and its summable solution (8.15). Summing all discarded tails proves (8.17). Finally, the decaying terminal regulator controls the zero-mode denominator and second moment, yielding (8.22) and the critical-mass identity. These uniform estimates supply the analytic hypotheses used by the nonlinear conjugacy in Section 7. $\square$

The theorem closes RG4 but does not remove or classify arbitrary spectator sectors. A spectator can carry an additional relevant field, conservation law, slow mode, or autonomous critical factor without violating any estimate above. Spectator exclusion, quotient completeness, and the associated counterexample gate are therefore logically separate and belong to Section 9. Their failure would activate the RG5 alternative, not reopen RG1–RG4.

9 Spectator-Sector Exclusion, Quotienting and Classification

9.1 Active–spectator decomposition and complete registry

The RG1–RG4 certificates concern the active scalar Model-A hierarchy. They do not, by themselves, exclude an additional degree of freedom that is invisible to the chosen filamentary observable but remains relevant to the full infrared theory. At every scale we therefore decompose the hierarchy as

$$\mathfrak{H}_j = (\mathfrak{A}_j, \mathfrak{S}_j, V_j^{\text{AS}}), \quad (9.1)$$

where $\mathfrak{A}_j$ contains the scalar order field, response field, mass, quartic coordinate, static polymers, and FDT-paired memory coordinate; $\mathfrak{S}_j$ is the complete candidate spectator sector; and V_j^{AS} contains all active–spectator couplings, including those generated under RG.

For every connected spectator component s , freeze the registry

$$\text{Spec}(s) = (m_s, \gamma_s, \mathcal{C}_s, \Sigma_s, \text{spec } L_s, V_s^{AS}, \mathfrak{a}_s, \Pi_s^{\text{obs}}, \mathcal{M}_s), \quad (9.2)$$

whose entries record, respectively, the static gap, relaxation rate, conservation laws, symmetry representation, linear RG spectrum, cross-couplings, anomaly/topological data, observable overlap, and induced memory tail. A microscopic degree of freedom absent from this registry is not eligible for quotienting.

The active response/noise completion is not a spectator: it is fixed by causality and FDT and is already included in $\mathfrak{A}_j$. Likewise, a field is not a spectator merely because the primary order observable has zero linear overlap with it.

9.2 Admissibility contracts S1–S7

Definition 9.1 (Quotient-admissible spectator). A registered component s is quotient-admissible if the following seven conditions hold uniformly along the tuned critical trajectory.

- S1 **No additional critical pole.** Its connected resolvent is analytic near $(\omega, k) = (0, 0)$, or every normalized spectator scaling field has a strictly negative RG exponent.
- S2 **Uniform static and dynamic decay.** There are $m_s, \gamma_s > 0$, independent of scale and terminal volume, such that static connected correlations decay on scale m_s^{-1} and dynamic correlations decay on scale γ_s^{-1} .
- S3 **Irrelevant cross-couplings.** Every symmetry-allowed active–spectator operator, including operators generated by blocking, has exponent at most $-\epsilon_s < 0$ after local normalization.
- S4 **Local Schur-complement closure.** Eliminating s produces only analytic local renormalizations and polymer/memory remainders contained in the RG4 stable ideal.
- S5 **No protected obstruction.** The sector has no anomaly, protected boundary mode, topological response, or superselection label retained by the comparison observables.
- S6 **Observable invisibility.** The frozen observable map has zero linear overlap with spectator scaling fields, while every nonlinear overlap is RG-irrelevant.
- S7 **Uniform determinant and reconstruction control.** Spectator free-energy and normalization factors are analytic across the active critical surface, and the induced reconstruction defect vanishes in the common hierarchy norm.

Conditions S1–S7 are conjunctive. In particular, a gap does not imply S5, microscopic decoupling does not imply S1, and observable invisibility does not imply S3.

9.3 RG-relevant quotient and congruence

Define

$$\mathfrak{H} \sim_{\text{sp}} \mathfrak{H}' \quad (9.3)$$

if the two hierarchies have identical active coordinates and differ only by spectator data satisfying S1–S7. Let

$$Q_{\text{sp}} : \mathfrak{H} \longmapsto [\mathfrak{H}]_{\text{sp}} \quad (9.4)$$

be the associated quotient map.

Proposition 9.2 (Spectator congruence). *On the S1–S7 admissible domain,*

$$\mathfrak{H} \sim_{\text{sp}} \mathfrak{H}' \implies \mathcal{R}\mathfrak{H} \sim_{\text{sp}} \mathcal{R}\mathfrak{H}'. \quad (9.5)$$

Consequently

$$\overline{\mathcal{R}}[\mathfrak{H}]_{\text{sp}} := [\mathcal{R}\mathfrak{H}]_{\text{sp}} \quad (9.6)$$

is independent of the representative.

Proof. S3 places all cross-coupling differences in a strictly contracting ideal. S4 maps their generated local parts into the already normalized active coordinates and their residual parts into the RG4 polymer/memory ideal. S1, S2, and S7 prevent a nonanalytic infrared or determinant term from being generated; S5 and S6 prevent loss of protected or observable data. Therefore one RG step preserves the equivalence relation, and induction gives the claim for the full sequence. $\square$

The quotient is thus a dynamical quotient, not a deletion performed only at the fixed point. Its kernel is the maximal registered stable ideal satisfying all seven contracts.

9.4 Gapped-spectator elimination

At quadratic order write the inverse propagator as

$$\mathcal{P}(\omega, k) = \begin{pmatrix} P_A(\omega, k) & C(\omega, k) \\ C(\omega, k)^* & P_S(\omega, k) \end{pmatrix}. \quad (9.7)$$

Under S1–S2, P_S^{-1} is analytic in a neighbourhood of the origin. Integrating out the spectator produces the Schur complement

$$P_A^{\text{eff}} = P_A - CP_S^{-1}C^*. \quad (9.8)$$

Its convergent derivative expansion has local mass, gradient, field, and kinetic pieces, which are absorbed into the frozen active normalizations. By S3, every remaining derivative has negative RG exponent.

For interacting spectators, exponential clustering from S2 bounds their connected cumulants by local trees. Expanding in the cross-couplings produces connected polymers satisfying

$$\|K_{j+1}^{(S)}\|_{j+1} \leq CL^{-\epsilon_s} \|K_j^{(S)}\|_j + C \|V_j^{AS}\|^2, \quad \|V_{j+1}^{AS}\| \leq CL^{-\epsilon_s} \|V_j^{AS}\|. \quad (9.9)$$

The resulting tail is summable and lies within the uniform truncation bound of Equation (8.17).

Theorem 9.3 (Admissible spectator elimination). *Eliminating an S1–S7 spectator preserves the active fixed point, its relevant and marginal multiplicities, the critical codimension, the dangerous quartic coordinate, the normalized active scaling functions, and the RG1–RG4 conjugacy class. The eliminated theory and the parent define the same element of the RG-relevant quotient.*

Proof. Equations (9.8) and (9.9) separate analytic local renormalizations from a summable stable remainder. S5–S7 ensure that the operation loses neither protected data nor retained observables. Applying Theorem 9.2 at each scale proves equality of the quotient trajectories and hence of their active infrared data. $\square$

9.5 Nondecoupling counterexample gate

The quotient must reject each of the following constructions.

C1: decoupled critical scalar. Tensor the certified active system with an independent massless scalar χ :

$$\mathfrak{H}' = \mathfrak{H}_A \otimes \mathfrak{H}_\chi, \quad V^{AS} = 0. \quad (9.10)$$

All active correlation functions remain unchanged, but the full fixed point is

$$\mathfrak{F}' = \mathfrak{F}_A \times \mathfrak{F}_\chi \quad (9.11)$$

and has additional scaling operators and multiplicities. Exact decoupling therefore does not authorize quotienting a critical spectator when full universality equivalence is claimed. This construction violates S1.

C2: conserved diffusive spectator. Add a conserved density n with a symmetry-allowed coupling $\lambda n \phi^2$. Its hydrodynamic pole persists at low frequency and may replace scalar Model-A dynamics by a coupled conserved-field dynamic class. This violates S1–S3 even if the static order observable is unchanged.

C3: protected or topological spectator. Tensor the active bulk with a gapped sector carrying a protected boundary response, anomaly, or superselection invariant. Bulk critical exponents may remain unchanged, while the full physical infrared data differ. This violates S5 and shows that a bulk gap alone is insufficient.

Proposition 9.4 (Nondecoupling gate). *If any registered component fails S1–S7, then it cannot be removed by Q_{sp} . In particular,*

$$\text{persistent active filamentary phase} \not\Rightarrow \text{full RG-universality equivalence} \quad (9.12)$$

without the spectator contract.

The three examples are constructive failure witnesses: C1 changes the fixed point product and spectral multiplicities; C2 changes the hydrodynamic content; C3 changes protected infrared data. None can be repaired by redefining nonuniversal metric factors.

9.6 Application to the frozen UMD comparison class

The frozen Gaussian/quartic modular UMD parent contains exactly one real scalar field and its response/noise completion. It contains no additional conserved density, Goldstone field, gauge mode, topological label, autonomous critical tensor factor, or unregistered slow field. Therefore

$$\text{Spec}_{\text{frozen}} = \emptyset, \quad Q_{\text{sp}} = \text{id}. \quad (9.13)$$

All S1–S7 contracts hold vacuously, while RG1–RG4 apply directly to the full frozen hierarchy rather than merely to a projected observable sector.

The same result extends to an explicitly enlarged parent containing massive, exponentially mixing local sectors only after every added component has been registered and verified against S1–S7. No unrestricted UMD model is covered by this extension statement.

9.7 RG5 theorem and induced branch labels

Theorem 9.5 (Spectator quotient and RG5 certification). *For the frozen weakly coupled scalar quartic modular UMD subclass in integer dimension $d \geq 5$, the complete spectator registry is empty and the RG-relevant spectator quotient is the identity. More generally, adjoining a uniformly gapped, exponentially mixing sector with strictly irrelevant cross-couplings and satisfying S1–S7 preserves the active RG1–RG4 class after the canonical quotient. Conversely, critical, hydrodynamic, protected, or relevantly coupled spectators cannot be quotiented without changing full universality data. Hence RG5 is certified precisely for the frozen scalar subclass and its S1–S7 admissible extensions.*

For the classification theorem of the next section, freeze the mutually exclusive labels

$$\begin{aligned}
\text{PURE-A : } & \text{Spec} = \emptyset, \\
\text{QUOT-A : } & \text{Spec} \neq \emptyset \text{ and every component satisfies S1–S7,} \\
\text{EXT-X : } & \text{at least one component is critical, hydrodynamic, protected,} \\
& \text{or relevantly coupled.}
\end{aligned} \tag{9.14}$$

Models labelled PURE-A and QUOT-A share the same scalar Model-A quotient. An EXT-X model belongs to a distinct candidate subclass and requires a branch-matched external reference theory. RG5 therefore closes the spectator gate without asserting that every UMD critical sequence is in the scalar Model-A universality class.

10 UMD Universality-Subclass Classification Theorem

10.1 Classification universe and outcome-blind invariants

Let $\mathfrak{C}_{T5}$ denote the set of thermodynamic UMD sequences that pass the entry, tuning, common-coarse-graining, full-sequence attraction, spectrum, nonlinear-field, irrelevant-tail, and spectator contracts established in Sections 2–9. Failure of an entry contract does not create a new positive universality subclass; it produces an explicit outside, failure, or unresolved label.

For $M \in \mathfrak{C}_{T5}$, define the operational fingerprint

$$\mathfrak{F}(M) = (d, G, n_{\text{op}}, \mathbf{C}, \mathbf{R}, \mathbf{A}, \mathbf{N}, \mathbf{L}, r, \mathbf{T}). \tag{10.1}$$

Here G is the order-parameter symmetry representation, n_{op} its component content, $\mathbf{C}$ the conservation matrix, $\mathbf{R}$ the reversibility/FDT flag, $\mathbf{A}$ the absorbing-state flag, $\mathbf{N}$ the noise/memory type, $\mathbf{L}$ the short- or long-range locality datum, r the tuning codimension, and $\mathbf{T}$ the gauge, topological, and anomaly registry.

The fingerprint is constructed before inspecting critical exponents or choosing an external reference label. It determines a candidate dynamical branch, but not by itself a universality

subclass. In particular,

$$\mathfrak{F}(M) = \mathfrak{F}(M') \not\Rightarrow \Sigma_{\text{RG}}(M) = \Sigma_{\text{RG}}(M'). \quad (10.2)$$

The failure of the converse is witnessed by multicritical fixed points, distinct short- and long-range fixed points, and identical active fingerprints carrying different nonquotient spectators.

10.2 Spectator index and normalized RG signature

The spectator analysis of Theorem 9.5 assigns

$$\mathfrak{S}(M) \in \{\text{PURE}, \text{QUOT}[S], \text{EXT}[\Sigma_S], \text{UNRESOLVED}\}. \quad (10.3)$$

The first label means that the spectator registry is empty. The second means that every registered component satisfies S1–S7 and is removed by the canonical RG quotient. The third retains at least one critical, hydrodynamic, protected, or relevantly coupled sector. The fourth records failure of spectator identifiability.

For a model passing RG1–RG5, define the complete normalized signature

$$\Sigma_{\text{RG}}(M) = [\mathfrak{F}(M), \mathfrak{S}(M), \mathcal{F}_*, DR|_{\mathcal{F}_*}, \mathcal{N}_*, \mathcal{D}, \mathcal{O}, r]_{\cong}. \quad (10.4)$$

The entries after $\mathfrak{S}$ are, respectively, the full-sequence fixed point, linearized operator with multiplicities and invariant subspaces, nonlinear normal form, dangerous-irrelevance ledger, normalized observable embedding, and critical codimension. The equivalence relation $\cong$ quotients only:

- (i) admissible analytic changes of scaling coordinates;
- (ii) redundant field, time, and noise normalizations;
- (iii) operator-basis changes inside equal symmetry sectors;
- (iv) S1–S7 spectator equivalence.

It does not quotient a relevant eigenvalue, critical pole, conservation law, protected label, or nontrivial dangerous coordinate.

Definition 10.1 (UMD universality subclass). Two RG1–RG5-certified UMD models lie in the same UMD universality subclass if and only if their normalized signatures (10.4) agree.

This definition is stronger than equality of critical exponents and weaker than microscopic isomorphism.

10.3 Deterministic branch selector

Define the selector $\mathcal{B}$ by the following frozen first-failure order:

G1 Entry gate. Failed admissibility or absence of a critical tuning gives $\text{OUT}[E]$.

G2 Fingerprint gate. Non-identifiable operational data give $\text{UNRESOLVED}[F]$.

G3 Spectator gate. Apply S1–S7 and assign PURE, QUOT, EXT, or UNRESOLVED[S].

G4 **RG gate.** A false RG1–RG5 requirement gives $\text{FAIL}[G_i]$; an unavailable proof object gives $\text{UNRESOLVED}[G_i]$.

G5 **Signature gate.** Normalize all certified fixed-point data and return $\text{U}[\Sigma_{\text{RG}}]$.

The first non-pass result terminates the selector. No survival outcome, measured exponent, preferred external theory, or desired classification label enters the branching rule.

10.4 Subclass registry

The branch registry induced by $\mathcal{B}$ is

Branch	Defining data	Consequence
$A_G^{\text{PURE}}(d)$	Scalar $\mathbb{Z}_2$, nonconserved, reversible/FDT, short-range dynamics; Gaussian fixed point; empty spectator registry	Certified frozen Gaussian Model-A UMD subclass
$A_G^{\text{QUOT}}(d; S)$	Same active data; every spectator satisfies S1–S7	Same quotient subclass as $A_G^{\text{PURE}}(d)$
$A_G^{\text{EXT}}(d; \Sigma_S)$	Same active fingerprint plus a nonquotient critical, hydrodynamic, or protected spectator	Distinct product or coupled subclass
$\text{U}[\Sigma]$	Any other fully identified fingerprint with certified RG signature	Separate fingerprint- and signature-indexed subclass
$\text{UNRESOLVED}[X]$	Fingerprint, spectator, or RG datum is not identifiable	No positive equivalence claim
$\text{FAIL}[G_i]$	A required RG gate is false	Counterexample or failure branch
$\text{OUT}[E]$	Entry assumptions fail	Outside the theorem domain

For the quartic modular parent constructed in Section 3,

$$\mathcal{B}(M_{\text{qUMD}}) = A_G^{\text{PURE}}(d), \quad d \geq 5, \quad z = 2, \quad r = 1, \quad y_u = 4 - d < 0. \quad (10.5)$$

An S1–S7-admissible extension satisfies

$$\mathcal{B}(M_{\text{qUMD}} \oplus S_{\text{adm}}) = A_G^{\text{QUOT}}(d; S_{\text{adm}}) \quad (10.6)$$

and has the same quotient signature as (10.5).

10.5 Branch completeness and noncollision

Theorem 10.2 (Disjoint exhaustive branch assignment). *For every proposed UMD critical sequence, the frozen selector returns exactly one element of*

$$\{\text{OUT}, \text{FAIL}, \text{UNRESOLVED}\} \cup \{\text{U}[\Sigma]\}. \quad (10.7)$$

On the RG1–RG5-certified domain, $\text{U}[\Sigma]$ is invariant under the common outcome-blind RG, admissible analytic reparameterization, redundant-operator removal, and S1–S7 spectator quotienting.

Proof. Every gate has mutually exclusive pass, fail, and unresolved outcomes, and the first non-pass outcome terminates the ordered selector. If all gates pass, normalization produces exactly one equivalence class $[\Sigma_{\text{RG}}]_{\cong}$. RG conjugacy preserves the fixed point, linear spectrum, nonlinear normal form, dangerous-variable ledger, and observable embedding up to the defining equivalence. By Theorem 9.2, admissible spectator quotienting preserves the same data. The assignment is therefore total and single-valued, and its positive output is RG-invariant. $\square$

Theorem 10.3 (Subclass noncollision). *Two certified models that differ in any nonredundant component of (10.4) cannot be assigned to the same subclass. In particular, an EXT spectator carrying an additional critical pole, a different relevant multiplicity, a conservation law, or a protected sector cannot collide with the $A_G^{\text{PURE/QUOT}}(d)$ subclass.*

Proof. Equality of subclasses is equality of normalized signatures. None of the listed differences belongs to the quotient relation $\cong$. Hence any such difference separates the equivalence classes. $\square$

10.6 Outcome-blind external-reference assignment

Classification inside UMD precedes external comparison. Define the candidate reference assignment map

$$\mathcal{R}_{\text{ext}} : \mathcal{U}[\Sigma] \mapsto \text{CandRef}[\Sigma] \quad (10.8)$$

by matching the complete preregistered fingerprint and spectator status, before comparing exponent values or scaling curves. For the certified scalar branches,

$$\begin{aligned} \mathcal{R}_{\text{ext}}(A_G^{\text{PURE}}(d)) &= \text{Model } A_G^{\text{PURE}}(d), \\ \mathcal{R}_{\text{ext}}(A_G^{\text{QUOT}}(d; S)) &= \text{Model } A_G^{\text{PURE}}(d) \quad \text{after } Q_{\text{sp}}. \end{aligned} \quad (10.9)$$

For $\text{EXT}[\Sigma_S]$, the assigned candidate must contain the same critical, conserved, hydrodynamic, or protected spectator signature:

$$\mathcal{R}_{\text{ext}}(A_G^{\text{EXT}}(d; \Sigma_S)) = \text{CandRef}[\text{Model } A_G(d) \times_V \Sigma_S]. \quad (10.10)$$

If no independently defined reference object with the required signature is available, the output is $\text{UNASSIGNED}[\Sigma]$, not the nearest reference class by exponent matching.

Proposition 10.4 (Reference-assignment firewall). *The map $\mathcal{R}_{\text{ext}}$ selects a comparison target but does not establish equivalence. A positive external-equivalence claim additionally requires a bidirectional scaling-field and observable dictionary, equality of normalized signatures, and quantitative reconstruction control.*

Thus Section 10 cannot use the name Model A as the proof of RG7. It only freezes the unique admissible target for the subsequent equivalence theorem.

10.7 RG6 certification and handoff

Theorem 10.5 (UMD subclass classification and RG6). *On the RG1–RG5-certified comparison universe, the fingerprint–spectator pair together with the normalized full RG signature defines a preregistered, exhaustive, nonoverlapping, RG-invariant UMD universality-subclass classification. The quartic modular parent lies in $A_G^{\text{PURE}}(d \geq 5)$; every S1–S7-admissible gapped extension lies*

in the corresponding $A_G^{\text{QUOT}}(d \geq 5; S)$ branch and has the same quotient signature. Nonquotient spectators, failed gates, unresolved data, and out-of-domain models remain separate. Hence RG6 is certified.

The result is an internal UMD classification theorem. It does not yet prove

$$A_G(d \geq 5)_{\text{UMD}} \equiv \text{Model } A_G(d \geq 5)_{\text{external}}. \quad (10.11)$$

That conclusion requires the RG7 comparison with the independently defined external object selected by (10.9). Filamentary persistence alone does not perform this selection; the complete RG1–RG6 signature does.

11 External Reference-Class Equivalence Theorem

11.1 Independently defined external reference object

The candidate selected by Equation (10.9) is now fixed independently of the UMD output. Let $A_{\text{ref}}(d)$ be the scalar stochastic dynamics

$$\partial_s \phi(x, s) = -\Gamma \frac{\delta \mathcal{H}}{\delta \phi(x, s)} + \zeta(x, s), \quad (11.1)$$

with static functional

$$\mathcal{H}[\phi] = \int_{\mathbb{R}^d} \left[\frac{1}{2} |\nabla \phi|^2 + \frac{t}{2} \phi^2 + \frac{u}{4!} \phi^4 - h \phi \right] d^d x \quad (11.2)$$

and equilibrium noise

$$\mathbb{E} \zeta(x, s) \zeta(x', s') = 2\Gamma T \delta(x - x') \delta(s - s'). \quad (11.3)$$

This is the short-range, scalar, $\mathbb{Z}_2$ -symmetric, nonconserved, reversible/FDT Model-A reference branch. It has no absorbing configuration and no nonquotient spectator. The Gaussian representative is obtained at $t = h = u = 0$ after canonical gradient, temperature, and time normalization.

The quartic coordinate remains in the enlarged scaling chart even though

$$y_u = 4 - d < 0. \quad (11.4)$$

It is not part of the Gaussian fixed-point action, but it is required for ordered-phase, zero-mode, and finite-size reconstruction. Thus the reference object is not defined by setting $u = 0$ in every physical observable.

11.2 Common normal-form chart and RG conjugacy

Let $\mathcal{X}_U^q$ be the RG5 spectator-quotiented UMD state space and $\mathcal{X}_A$ the corresponding reference dynamic-functional state space. RG2–RG4 provide local charts

$$\Psi_U : \mathcal{N}_U \subset \mathcal{X}_U^q \longrightarrow \mathcal{Y}, \quad \Psi_A : \mathcal{N}_A \subset \mathcal{X}_A \longrightarrow \mathcal{Y}, \quad (11.5)$$

where

$$\mathcal{Y} = \{(t, h, u, \Gamma, Z_\phi, Z_{\tilde{\phi}}, \mathbf{w}, \mathbf{m})\}. \quad (11.6)$$

The coordinates $\mathbf{w}$ and $\mathbf{m}$ contain, respectively, the static polymer remainder and FDT-paired nonlocal memory/noise remainder. Redundant normalization directions have already been quotiented.

Both theories are intertwined with the common normal form

$$\mathcal{N}_L(y) = (L^2 t, L^{(d+2)/2} h, L^{4-d} u, \Gamma, Z_\phi, Z_{\tilde{\phi}}, K_L \mathbf{w}, K_L^m \mathbf{m}) + \mathcal{Q}(y), \quad (11.7)$$

where

$$\max\{\|K_L\|, \|K_L^m\|\} \leq \rho < 1, \quad \|\mathcal{Q}(y)\| \leq C \|y\|^{1+\epsilon}, \quad (11.8)$$

and the normalized nonlinear germ $\mathcal{Q}$ is the same on both sides.

Define

$$\mathfrak{C} := \Psi_A^{-1} \circ \Psi_U, \quad \mathfrak{C}^{-1} = \Psi_U^{-1} \circ \Psi_A. \quad (11.9)$$

Then

$$\mathfrak{C} \circ \mathcal{R}_U = \mathcal{R}_A \circ \mathfrak{C}. \quad (11.10)$$

This is a local analytic equivalence between RG quotient neighbourhoods. It is not a bijection between microscopic configurations, which coarse graining cannot provide.

11.3 Bidirectional scaling-field dictionary

After the common kinetic, noise, and gradient normalization, the forward dictionary has the expansion

$$\begin{aligned} t_A &= a_t t_U + O(\|y_U\|^2), & h_A &= a_h h_U + O(\|y_U\|^2), \\ u_A &= a_u u_U + O(\|y_U\|^2), & \Gamma_A &= a_\Gamma \Gamma_U + O(\|y_U\|^2), \\ \mathbf{w}_A &= B_w \mathbf{w}_U + O(\|y_U\|^2), & \mathbf{m}_A &= B_m \mathbf{m}_U + O(\|y_U\|^2). \end{aligned} \quad (11.11)$$

Here a_t, a_h, a_u, a_Γ are nonzero nonuniversal metric factors, while B_w and B_m are boundedly invertible on the retained irrelevant charts. The analytic inverse-function theorem supplies the reverse dictionary.

In particular,

$$\mathfrak{C}(E_U^{\text{rel}}) = E_A^{\text{rel}}, \quad (11.12)$$

$$\mathfrak{C}(E_U^{\text{dang}}) = E_A^{\text{dang}}, \quad (11.13)$$

$$\mathfrak{C}(E_U^{\text{irr}}) = E_A^{\text{irr}}. \quad (11.14)$$

No step identifies the dangerous quartic field with a regular irrelevant coordinate or sets it to zero before observable reconstruction.

11.4 Bidirectional observable dictionary

Let O_U be the normalized UMD scalar slow mode and $\tilde{O}_U$ its response field. There exist positive metric factors Z_O, Z_s such that

$$\phi_A = Z_O^{-1/2} O_U, \quad \tilde{\phi}_A = Z_O^{1/2} Z_s^{-1} \tilde{O}_U, \quad s_A = Z_s s_U. \quad (11.15)$$

The induced dictionary for basic observables is

$$\begin{aligned} C_A(x, s) &= Z_O^{-1} C_U(x, Z_s^{-1} s), \\ R_A(x, s) &= Z_s^{-1} R_U(x, Z_s^{-1} s), \\ \chi_A &= Z_O^{-1} \chi_U, \\ S_A(k, \omega) &= Z_O^{-1} Z_s S_U(k, Z_s \omega). \end{aligned} \tag{11.16}$$

Composite operators are transported by renormalized normal ordering and the invertible operator-mixing matrix within each equal-symmetry sector. The inverse dictionary uses reciprocal metric factors and the inverse mixing matrix.

The map preserves causality, $\mathbb{Z}_2$ parity, nonconservation of the order parameter, equilibrium FDT, and the absence of an absorbing-state constraint. Individual metric factors and dimensional amplitudes are not claimed to be universal. Dimensionless scaling functions, exponent data, and normalized amplitude ratios are the invariant objects.

For observables singular in u , the dictionary acts first on

$$M = \sqrt{u} m, \quad H = \sqrt{u} h, \quad \Theta = uT, \tag{11.17}$$

and physical variables are reconstructed only afterwards. This preserves the dangerous amplitudes established in Sections 7 and 8.

11.5 Quantitative reconstruction control

Let $\mathcal{A}_U^{(m)}$ and $\mathcal{A}_A^{(m)}$ be the quotient algebras of renormalized local/response observables of degree at most m . Define

$$\mathfrak{D}_{U \rightarrow A}^{(m)} := (\mathfrak{C}^{-1})^*, \quad \mathfrak{D}_{A \rightarrow U}^{(m)} := \mathfrak{C}^*. \tag{11.18}$$

For each fixed m and compact scaled momentum–frequency window, there are $C_m < \infty$ and $0 < \rho < 1$ such that

$$\left\| \mathfrak{D}_{A \rightarrow U}^{(m)} \mathfrak{D}_{U \rightarrow A}^{(m)} O - O \right\|_{j,m} \leq C_m \rho^j \|O\|_{j,m}, \tag{11.19}$$

and

$$\left\| \mathfrak{D}_{U \rightarrow A}^{(m)} \mathfrak{D}_{A \rightarrow U}^{(m)} O' - O' \right\|_{j,m} \leq C_m \rho^j \|O'\|_{j,m}. \tag{11.20}$$

The defect consists only of certified irrelevant and redundant operators. For dangerous observables, the norm retains u_j and the blown-up coordinates; no uniformity at $u = 0$ is asserted before specifying the thermodynamic or finite-size limit.

At the level of RG trajectories, the same contraction yields

$$\|\Psi_A \mathcal{R}_A^n \mathfrak{C}(x) - \Psi_U \mathcal{R}_U^n x\|_Y \leq C \rho^n (\|\mathbf{w}(x)\| + \|\mathbf{m}(x)\|). \tag{11.21}$$

In the exact frozen common normal-form chart, retained scaling coordinates agree and only the contracting remainder appears on the right.

11.6 Normalized-signature equality

The equality test required by Theorem 10.4 is componentwise:

Signature component	UMD quotient	External reference
Dimension/order field	Scalar in $d \geq 5$	Scalar in $d \geq 5$
Symmetry	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Conservation	Nonconserved	Nonconserved
Equilibrium structure	Detailed balance/FDT	Detailed balance/FDT
Absorbing constraint	Absent	Absent
Fixed point	Gaussian OU	Gaussian OU
Relevant spectrum	One mass direction, $y_t = 2$	One mass direction, $y_t = 2$
Magnetic field	$y_h = (d+2)/2$ before frozen symmetry restriction	$y_h = (d+2)/2$ before frozen symmetry restriction
Dangerous field	Quartic, $y_u = 4 - d$	Quartic, $y_u = 4 - d$
Dynamic exponent	$z = 2$	$z = 2$
Nonlinear germ	$\mathcal{N}_L$	The same $\mathcal{N}_L$
Observable embedding	(11.15)	Inverse of (11.15)
Spectator index	PURE or S1–S7 quotient	PURE
Remainder control	Summable polymer/memory tail	Summable polymer/memory tail

Therefore

$$\Sigma_{\text{RG}}(Q_{\text{sp}}M_{\text{qUMD}}) = \Sigma_{\text{RG}}(\mathbf{A}_{\text{ref}}(d)). \quad (11.22)$$

The Gaussian leading critical data are consequently

$$\eta = 0, \quad \nu = \frac{1}{2}, \quad z = 2, \quad (11.23)$$

together with the mean-field thermodynamic exponent relations already derived from the dangerous-variable chart. Equality of these exponents is a corollary of (11.22), not its proof.

11.7 External-equivalence theorem and RG7

Theorem 11.1 (External Gaussian Model-A equivalence). *For the frozen weakly coupled scalar, short-range, reversible/FDT, $\mathbb{Z}_2$ -symmetric modular UMD subclass in integer dimension $d \geq 5$, and for every S1–S7-admissible extension after the canonical spectator quotient, the RG state is locally analytically conjugate to the independently defined external Gaussian Model-A reference $\mathbf{A}_{\text{ref}}(d)$. The conjugacy is bidirectional on the RG quotient, preserves relevant, dangerous, and irrelevant invariant subspaces, transports the scaling-field and observable dictionaries, and has the scale-decaying reconstruction defects (11.19)–(11.21). The complete normalized RG signatures agree as in (11.22). Hence RG7 is certified.*

Proof. RG1 fixes the common outcome-blind hierarchy and topology; RG2 gives unique full-sequence attraction; RG3 identifies the linear spectra and multiplicities; RG4 supplies the common nonlinear germ and uniform irrelevant-tail control; RG5 establishes the legitimate spectator quotient; and RG6 assigns the reference target without using output data. The common charts then define (11.9), and their intertwining identities give (11.10). The inverse-function theorem gives the bidirectional scaling dictionary, while pullback gives the observable dictionary. RG4 contraction proves the reconstruction bounds. The componentwise ledger establishes equality of the normalized signatures. $\square$

The theorem does not cover $d = 4$, $d < 4$, conserved or nonequilibrium branches, long-range interactions, absorbing-state dynamics, or EXT spectators without a branch-matched external theory. It establishes external equivalence of one rigorously delimited UMD subclass, not a collapse of all UMD critical sequences into one universality class.

12 Final T5 Theorem, Domain of Validity and Failure Alternatives

12.1 Positive theorem domain

Let $\mathfrak{D}_{T5}^+(d)$, with integer $d \geq 5$, be the class of thermodynamic UMD sequences satisfying the following contracts. These contracts are fixed before the final universality conclusion is evaluated.

Thermodynamic and carrier contracts.

- H_1 The finite UMD objects admit the frozen thermodynamic embedding, uniform quasi-locality, and microscopic consistency bounds.
- H_2 The outcome-blind filamentary carrier exists throughout the certified phase interval and satisfies the T1–T4 persistence, crossing-density, and mark-stability conclusions.
- H_3 Carrier extraction and its local observables satisfy the stabilization, local-weak transfer, uniform block-contiguity, and dependence-defect bounds.

Critical fingerprint contracts.

- H_4 The active registry contains one real $\mathbb{Z}_2$ -odd scalar slow mode.
- H_5 The active order field is nonconserved and purely relaxational.
- H_6 Detailed balance and FDT hold, and no absorbing-state constraint is present.
- H_7 The active kernel is short-ranged and the critical tuning manifold has codimension $r = 1$.
- H_8 The spatial dimension satisfies $d > 4$; in the lattice realization used here, $d \geq 5$.

RG and spectator contracts.

- H_9 RG1: a common outcome-blind UMD/reference coarse-graining map and Banach topology exist.
- H_{10} RG2: the tuned full sequence converges uniquely to the Gaussian Ornstein–Uhlenbeck fixed point.
- H_{11} RG3: the relevant, marginal, dangerous, and irrelevant spectra agree with the frozen Gaussian Model-A spectrum.
- H_{12} RG4: the nonlinear scaling-field conjugacy, dangerous quartic reconstruction, dynamic locality, and uniform irrelevant-tail estimates hold.
- H_{13} RG5: the spectator registry is PURE, or QUOT with S1–S7 certified.

- H_{14} RG6: the sequence is uniquely assigned to $A_G^{\text{PURE}}(d)$ or $A_G^{\text{QUOT}}(d; S)$.
- H_{15} RG7: the external scaling-field/observable dictionaries, normalized-signature equality, and quotient reconstruction estimates hold.

The exact positive domain is

$$\mathfrak{D}_{\text{T5}}^+(d) := \bigcap_{i=1}^{15} \mathfrak{D}(H_i). \quad (12.1)$$

12.2 Domain-of-validity lattice

Define the nested domains

$$\begin{aligned} \mathfrak{D}_0 &= \{\text{all proposed UMD thermodynamic sequences}\}, \\ \mathfrak{D}_1 &= \mathfrak{D}_0 \cap \mathfrak{D}(H_1), \\ \mathfrak{D}_2 &= \mathfrak{D}_1 \cap \mathfrak{D}(H_2, H_3), \\ \mathfrak{D}_3 &= \mathfrak{D}_2 \cap \mathfrak{D}(H_4, \dots, H_8), \\ \mathfrak{D}_4 &= \mathfrak{D}_3 \cap \mathfrak{D}(H_9, \dots, H_{12}), \\ \mathfrak{D}_5 &= \mathfrak{D}_4 \cap \mathfrak{D}(H_{13}, H_{14}), \\ \mathfrak{D}_6 &= \mathfrak{D}_5 \cap \mathfrak{D}(H_{15}) = \mathfrak{D}_{\text{T5}}^+(d). \end{aligned} \quad (12.2)$$

Thus

$$\mathfrak{D}_6 \subseteq \mathfrak{D}_5 \subseteq \dots \subseteq \mathfrak{D}_1 \subseteq \mathfrak{D}_0. \quad (12.3)$$

Membership in a lower domain does not imply membership in a higher one. In particular,

$$M \in \mathfrak{D}_2 \not\Rightarrow M \in \mathfrak{D}_3. \quad (12.4)$$

Equation (12.4) is the formal statement that filamentary persistence alone does not determine a dynamic universality class.

12.3 Restricted positive T5 theorem

Theorem 12.1 (Restricted positive T5 equivalence). *Let $d > 4$ and let $M \in \mathfrak{D}_{\text{T5}}^+(d)$. Let $A_{\text{ref}}(d)$ be the independently fixed scalar, short-range, equilibrium Model-A reference of Section 11.1. Then:*

- (i) *M possesses the persistent asymptotic filamentary phase certified by T1–T4;*
- (ii) *after the canonical S1–S7 spectator quotient,*

$$Q_{\text{sp}}M \in A_G(d); \quad (12.5)$$

- (iii) *the quotient and the external reference have identical normalized RG signatures:*

$$\Sigma_{\text{RG}}(Q_{\text{sp}}M) = \Sigma_{\text{RG}}(A_{\text{ref}}(d)); \quad (12.6)$$

- (iv) *there is a local analytic, bidirectional scaling-field conjugacy*

$$\mathfrak{C} : \mathcal{N}_{\text{U}} \subset \mathcal{X}_{\text{U}}^q \longleftrightarrow \mathcal{X}_{\text{A}} \supset \mathcal{N}_{\text{A}}; \quad (12.7)$$

(v) the induced observable dictionaries are bidirectional on the RG quotient and, for every fixed observable degree m , satisfy

$$\left\| \mathfrak{D}_{A \rightarrow U}^{(m)} \mathfrak{D}_{U \rightarrow A}^{(m)} O - O \right\|_{j,m} \leq C_m \rho^j \|O\|_{j,m}, \quad 0 < \rho < 1; \quad (12.8)$$

(vi) the UMD quotient and the external reference belong to the same Gaussian Model-A RG universality class:

$$\boxed{Q_{\text{sp}} M \simeq_{\text{RG}} A_{\text{ref}}(d)}. \quad (12.9)$$

Proof. H_1 – H_3 provide the infinite-volume carrier and the T1–T4 physical observable conclusions. Contracts H_4 – H_8 place the active sector in the scalar, short-range, equilibrium Model-A comparison branch. RG1 gives the common operational hierarchy. RG2 gives unique full-sequence attraction to the common Gaussian fixed point. RG3 identifies the linear spectrum and multiplicities. RG4 supplies the common nonlinear normal form, retains the dangerously irrelevant quartic coordinate, and controls the uniform irrelevant tail. RG5 removes exactly the admissible spectators and rejects nondecoupling sectors. RG6 gives a unique noncolliding subclass assignment. Finally, RG7 supplies the external conjugacy, bidirectional observable dictionary, normalized-signature equality, and reconstruction estimate. Composing these certificates gives (i)–(vi). $\square$

Equivalently, within (12.1),

$$\boxed{\begin{aligned} &\text{persistent asymptotic UMD filamentary phase} \\ &+ \text{scalar nonconserved equilibrium fingerprint} \\ &\quad + \text{RG1–RG7 certificates} \\ &\quad + \text{PURE/QUOT spectator condition} \\ &\implies \text{external Gaussian Model-A RG equivalence.} \end{aligned}} \quad (12.10)$$

12.4 The valid equivalence form and universal conclusions

The theorem does not claim that an arbitrary external Model-A system must possess a microscopic UMD filamentary carrier. That converse would require a reference-to-UMD realization theorem not contained in T5. The valid biconditional is internal to the frozen comparison domain:

$$\boxed{\begin{aligned} &Q_{\text{sp}} M \simeq_{\text{RG}} A_{\text{ref}}(d) \\ &\iff \Sigma_{\text{RG}}(Q_{\text{sp}} M) = \Sigma_{\text{RG}}(A_{\text{ref}}(d)), \end{aligned}} \quad (12.11)$$

provided H_1 and H_4 – H_{15} already hold. Filamentary persistence is an independently certified UMD phase property; it is not a complete invariant of the external RG class.

Inside the positive domain, the common leading data include

$$\eta = 0, \quad \nu = \frac{1}{2}, \quad z = 2, \quad y_u = 4 - d < 0, \quad (12.12)$$

together with the Gaussian/mean-field exponent relations, normalized dynamic scaling functions, and the amplitude ratios established in Section 7. The quartic coordinate remains in every finite-size or zero-mode expression for which it is dangerously irrelevant.

The theorem identifies fixed-point and basin membership, spectral multiplicities, the nonlinear scaling-field normal form, universal exponent data, dimensionless scaling functions, normalized amplitude ratios, response/correlation correspondence, and quotient-observable reconstruction. It does not identify nonuniversal metric factors or microscopic configurations.

12.5 First-failure alternatives

Theorem 12.2 (Exhaustive first-failure adjudication). *Every proposed UMD thermodynamic sequence receives exactly one first outcome in the following ordered ledger.*

Code	First failed contract	Scientific conclusion
F_0	H_1	No admissible thermodynamic UMD object.
F_1	H_2-H_3	Infinite filamentary carrier or its persistence is not established.
F_2	H_4-H_8	The carrier may persist, but the frozen scalar Model-A fingerprint is absent.
F_3	$RG1/H_9$	No common outcome-blind coarse-graining comparison.
F_4	$RG2/H_{10}$	No unique full-sequence Gaussian attraction.
F_5	$RG3-RG4/H_{11}-H_{12}$	The spectrum, nonlinear scaling fields, or uniform remainder control differ.
F_6	$RG5/H_{13}$	A critical, hydrodynamic, protected, or relevantly coupled spectator survives.
F_7	$RG6/H_{14}$	The subclass is unresolved or belongs to another certified branch.
F_8	$RG7/H_{15}$	An internal UMD subclass exists, but external equivalence is not established.
P	H_1-H_{15} pass	The restricted positive T5 theorem applies.

Proof. The contracts are finite and ordered. Adjudication stops at the first false or unresolved contract, so two different first-failure codes cannot apply to the same sequence. If no failure occurs, all H_1-H_{15} hold and the output is P. The ledger is therefore mutually exclusive and exhaustive. $\square$

A failure code does not automatically assert absence of criticality or universality. It states that the specific conclusion (12.9) is not licensed. Several failure branches define different universality subclasses and separate proof programmes.

12.6 Counterexample boundary

The unrestricted implication is blocked by explicit mechanisms including:

- (1) a persistent filamentary carrier with a conserved active order field;
- (2) the same active scalar sector with a massless diffusive spectator;
- (3) a protected topological or anomalous spectator;
- (4) FDT-breaking noise or nonlocal memory that is not RG-irrelevant;

- (5) an absorbing-state constraint;
- (6) a long-range kernel outside the frozen short-range basin;
- (7) a non-Gaussian or multicritical fixed point;
- (8) $d \leq 4$, including the marginal $d = 4$ branch;
- (9) subsequence-dependent RG limits or failure of compactness.

Therefore

$$\boxed{\text{persistent filamentary phase} \not\Rightarrow \text{full external universality-class equivalence} \\ \text{without } H_4\text{--}H_{15}} \quad (12.13)$$

12.7 Frozen claim hierarchy and maximality

The final article distinguishes four claim tiers:

Tier I: Carrier.

Persistent infinite-volume filamentary structure and its physical crossing/mark observables.

Tier II: Fingerprint.

Identification of the scalar, nonconserved, equilibrium, short-range branch.

Tier III: Internal UMD classification.

RG1–RG6 and membership in the PURE/QUOT subclass.

Tier IV: External equivalence.

RG7 and equivalence to the independently fixed continuum Model-A reference.

Tier IV implies Tier III in the frozen comparison category. Tier I alone implies neither Tier II nor Tier IV.

The domain (12.1) is maximal relative to the proof objects used here: deleting any of $H_9\text{--}H_{15}$ removes a necessary component of the RG-equivalence definition or proof; deleting $H_4\text{--}H_8$ admits explicit alternative dynamic branches; deleting $H_2\text{--}H_3$ removes the filamentary-phase premise. This is proof-relative maximality, not a claim that no broader theorem can be proved with additional objects.

12.8 Section verdict

The exact proof-or-counterexample resolution is

$$\boxed{\text{T5}_{\text{AG}, d>4}^+ = \text{PROVED} \quad \text{on } \mathfrak{D}_{\text{T5}}^+(d)} \quad (12.14)$$

and

$$\boxed{\text{T5}_{\text{unrestricted}} = \text{FALSE.}} \quad (12.15)$$

The second statement rejects the implication quantified over unrestricted UMD sequences; it does not negate the positive theorem on the certified subclass. Together, the two verdicts give

the exact resolution requested by the T5 programme:

$$\boxed{\begin{array}{l} \text{Carrier} \longrightarrow \text{Fingerprint} \longrightarrow \text{RG1--RG6 classification} \\ \longrightarrow \text{RG7 external equivalence} \longrightarrow \text{T5}_{\text{AG}, d>4}^+ \end{array}} \quad (12.16)$$

13 Outlook

13.1 The proved anchor and the extension rule

The present article establishes the rigorously delimited anchor

$$\text{T5}_{\text{AG}, d>4}^+ \quad (13.1)$$

for the scalar, short-range, nonconserved, reversible/FDT, PURE/QUOT UMD subclass. Every extension must preserve the methodological separation

$$\boxed{\text{filamentary persistence is a phase premise, not a universality classifier.}} \quad (13.2)$$

Changing a dimension, conservation law, noise structure, or spectator content changes the fingerprint and therefore requires a new reference object and a new RG1–RG7 certificate. None of the programmes below is counted as part of Theorem 12.1.

13.2 The marginal branch at $d = 4$

At the upper critical dimension,

$$y_u = 4 - d = 0, \quad (13.3)$$

so the strict quartic contraction used above disappears. The expected leading normal form is

$$u_{j+1} = u_j - \beta_4 u_j^2 + O(u_j^3), \quad \beta_4 > 0, \quad u_j \sim \frac{1}{\beta_4 j}. \quad (13.4)$$

Consequently an exponential estimate $C\rho^j$ cannot be carried across $d = 4$ by continuity. It must be replaced by a marginal chart and explicit logarithmic control.

The required proof package is:

MARG4-1 a common UMD/reference marginal chart with matching universal beta-function coefficients;

MARG4-2 full-sequence critical-mass tuning with $u_j \downarrow 0$;

MARG4-3 static and dynamic polymer control relative to the marginal trajectory;

MARG4-4 logarithmically corrected response, correlation, and susceptibility dictionaries;

MARG4-5 zero-mode and finite-size reconstruction retaining the running quartic coupling;

MARG4-6 a polynomial/logarithmic round-trip defect replacing $C\rho^j$.

A positive theorem requires equality of the logarithmic exponents and normalized scaling functions. Mean-field power laws alone are insufficient. A different universal beta coefficient, nonsummable memory term, or different logarithmic correction produces a distinct marginal subclass.

13.3 Interacting branches below four dimensions

For $d < 4$,

$$y_u = 4 - d > 0, \quad (13.5)$$

and the quartic field is relevant at the Gaussian point. The target is an interacting Wilson–Fisher-type fixed point

$$\mathcal{F}_*^{\text{WF}} \neq \mathcal{F}_*^{\text{G}}. \quad (13.6)$$

The exponents, including η , ν , and the dynamic correction to z , must be outputs of the constructed fixed point rather than preregistered targets.

The corresponding proof package comprises:

- WF-1 existence of an interacting static UMD fixed point;
- WF-2 an FDT-compatible dynamic response extension;
- WF-3 tightness and unique full-sequence attraction in an interacting polymer norm;
- WF-4 the complete relevant/marginal spectrum and tuning codimension;
- WF-5 nonlinear scaling fields and operator-mixing control;
- WF-6 bidirectional observable reconstruction and external equivalence;
- WF-7 a proof-or-counterexample gate for strong-coupling, competing-fixed- point, or runaway behaviour.

The nearest controlled target is a $4 - \epsilon$ regime or a rigorously defined long-range surrogate with a small parameter. A direct three-dimensional theorem is a later and substantially harder objective.

13.4 Conserved and mode-coupled branches

Conservation changes the infrared pole and is part of the fingerprint. A Model-B-type reference has the schematic form

$$\partial_s \phi = \Gamma \Delta \frac{\delta \mathcal{H}}{\delta \phi} + \nabla \cdot \xi, \quad (13.7)$$

with conserved noise and Gaussian rate $\omega \sim k^4$, rather than the Model-A rate k^2 . The candidate leading value is therefore $z_B = 4$ before anomalous corrections. A positive UMD theorem needs an exact continuity equation, momentum-sector coercivity, a conserved-noise FDT dictionary, and a Model-B-specific spectrum.

If a nonconserved order field couples to a conserved scalar density, the active state is enlarged to

$$(\phi, e), \quad V_{\text{mix}} = \gamma \phi^2 e. \quad (13.8)$$

The density cannot be quotiented before the eigenvalue of γ is decided. The admissible alternatives are:

- (i) irrelevant coupling and asymptotic return to Model A;
- (ii) a stable coupled Model-C-type fixed point;
- (iii) crossover without unique full-sequence attraction;

(iv) a first-order or noncritical alternative.

The proper theorem targets are therefore a family

$$\text{T5}_B^+(d), \quad \text{T5}_C^+(d), \quad \dots, \quad (13.9)$$

indexed by the complete conservation and mode-coupling fingerprint [1].

13.5 Nonequilibrium and FDT-breaking branches

Without detailed balance, correlation and response are independent objects. The equilibrium FDT dictionary cannot be assumed. The minimal enlarged fingerprint is

$$\mathfrak{F}_{\text{NE}} = (G, C, A, N, \text{FDT}, J_{\text{ss}}, M, L), \quad (13.10)$$

where J_{ss} records stationary probability currents and M records coloured-noise and memory structure.

Three distinct RG outcomes must remain open:

Emergent equilibrium.

Every FDT-breaking coupling is strictly irrelevant and the flow returns to an equilibrium external class.

Nonequilibrium fixed point.

At least one FDT-breaking invariant survives and defines a new class.

No stationary scaling limit.

Ageing, limit cycles, runaway flow, or nonunique subsequential limits prevent a T5 equivalence theorem.

The necessary gates include independent response/correlation tightness, a causal memory norm, an entropy-production scaling field, positivity and noise-covariance control, a nonequilibrium spectator registry, and an external reference frozen before final exponent data are inspected.

13.6 Controlled EXT sectors

RG5 correctly excludes nonquotient spectators from the present theorem, but some EXT branches may admit separate classifications.

Product fixed points. If active-spectator couplings vanish under RG while the spectator remains critical, the limit may be

$$\mathcal{F}_*^{\text{active}} \times \mathcal{F}_*^{\text{spectator}}. \quad (13.11)$$

This is not the scalar Model-A class. Its signature contains both spectra and the tensor-product observable algebra.

Coupled fixed points. If g_{AS} is relevant or marginal, one must construct the joint flow

$$(\beta_A, \beta_S, \beta_{AS}) \quad (13.12)$$

and determine whether it has a stable coupled fixed point. Integrating out the spectator before this decision is forbidden.

Hydrodynamic and protected sectors. Hydrodynamic poles require a joint dynamic theory. Topological, anomalous, or superselection data require invariant matching beyond local bulk correlations. Such sectors may be invisible to the filamentary carrier and still be decisive for full external equivalence.

An EXT branch can enter a positive theorem only after:

- (1) the full joint fixed point is identified;
- (2) every cross-coupling eigenvalue is certified;
- (3) the enlarged observable algebra is reconstructible;
- (4) protected and anomalous data match the chosen external reference;
- (5) the result is labelled as a new subclass, never as a QUOT extension.

13.7 The UMD Universality Atlas

The long-term objective is a certified classifier

$$\mathcal{U}_{\text{UMD}} : M \mapsto (\mathfrak{F}, \mathfrak{S}, \Sigma_{\text{RG}}, \text{Reference}, \text{Verdict}). \quad (13.13)$$

The atlas should include Gaussian and interacting nonconserved equilibrium branches; conserved and mode-coupled branches; absorbing-state and long-range branches; emergent-equilibrium and genuinely nonequilibrium branches; product and coupled EXT branches; protected topological/anomalous branches; and explicit unresolved, runaway, and out-of-domain alternatives.

Atlas completeness means that every admissible input receives either a certified subclass or an explicit OUT/FAIL/UNRESOLVED label. Similar exponent estimates alone must never merge two entries. The target is

$$\boxed{\{\text{UMD critical sequences}\} \longrightarrow \{\text{fingerprint- and spectator-indexed external classes}\}.} \quad (13.14)$$

13.8 Proof-oriented numerical programme and priorities

Future computations should construct proof objects rather than replace them. Useful certified targets include interval enclosures of beta coefficients and RG eigenvalues, verified continuation of fixed-point branches, finite-volume crossover bounds, certified response/correlation amplitude ratios, and automated first-failure classification under a frozen selector. Each target must be preregistered, outcome-blind, and connected to an analytic theorem by an explicit contract.

The recommended proof-continuity order is:

1. the marginal Gaussian extension at $d = 4$;
2. the conserved Model-B branch;
3. controlled product EXT sectors;
4. near-four-dimensional interacting Model A;

5. Model C and coupled EXT fixed points;
6. nonequilibrium/FDT-breaking classification;
7. direct three-dimensional interacting UMD equivalence.

This order ranks continuity with the present proof, not physical importance. Nearby constructive RG frameworks provide possible ingredients, but do not by themselves prove any UMD extension [16].

13.9 Non-negotiable boundaries and outlook verdict

The following shortcuts remain forbidden:

- extrapolating the $d > 4$ theorem to $d = 4$ without logarithmic control;
- extrapolating it to $d < 4$ without an interacting fixed point;
- treating a conserved field as a quotientable spectator;
- inferring FDT from matching exponents;
- deleting a critical EXT sector because it is invisible to the filamentary carrier;
- replacing full-sequence convergence by a finite-size fit;
- using carrier geometry as the sole universality classifier.

The present result is therefore one certified anchor rather than a completed atlas:

$$\boxed{\text{T5}_{\text{A}_G, d>4}^+ \longrightarrow \text{a branchwise certified UMD Universality Atlas.}} \quad (13.15)$$

The modular separation of carrier geometry, active fingerprint, spectator content, RG fixed point, and external dictionary makes this expansion possible without weakening the theorem already proved.

14 Discussion

14.1 What the theorem adds

The main result closes a logical gap that cannot be closed by carrier percolation, finite-size scaling, symmetry labels, or exponent matching alone. Earlier parts of the UMD programme established an infinite-volume filamentary carrier, physical persistence observables, and the strict boundary

$$\text{persistent filamentary phase} \not\Rightarrow \text{full universality-class equivalence.} \quad (14.1)$$

The present work supplies the missing RG bridge. On the certified scalar, short-range, nonconserved, reversible/FDT, PURE/QUOT branch in $d > 4$, it proves

$$\text{T5}_{\text{A}_G, d>4}^+ : \quad Q_{\text{sp}} M \simeq_{\text{RG}} \text{A}_{\text{ref}}(d). \quad (14.2)$$

The advance is not the observation of another mean-field exponent. It is a complete chain from a thermodynamic UMD sequence to an independently fixed external dynamic universality class.

14.2 Mathematical significance

A branch-entry theorem. The work does not redefine the Hohenberg–Halperin Model-A class [1]. It proves that one specified UMD subclass enters a fixed external representative of that class:

$$\text{external class definition} \neq \text{UMD branch-entry theorem.} \quad (14.3)$$

The external class is characterized independently by symmetry, conservation, equilibrium structure, locality, and its critical RG fixed point. The UMD theorem supplies the model-specific verification that these data arise from the thermodynamic modular sequence.

Full-sequence attraction. Universality is formulated through convergence of the full tuned RG trajectory. A favorable subsequence, a long crossover plateau, or an apparently stable finite-size fit is insufficient. This rules out false identification caused by oscillatory flows, multiple basins, or subsequence-dependent limits.

Equivalence beyond exponents. The invariant is the normalized signature

$$\Sigma_{\text{RG}} = [\mathfrak{F}, \mathfrak{S}, \mathcal{F}_*, D\mathcal{R}|_{\mathcal{F}_*}, \mathcal{N}_*, \mathcal{D}, \mathcal{O}, r]_{\cong}. \quad (14.4)$$

It contains the fixed point, spectral multiplicities, nonlinear normal form, dangerous variables, spectator quotient, and observable embedding. Equality of selected critical exponents is a consequence of signature equality, not its substitute.

Controlled irreversibility. Coarse graining is many-to-one. The strongest meaningful reverse statement is therefore a reconstruction theorem on the quotient observable algebra:

$$\left\| \mathfrak{D}_{\text{A} \rightarrow \text{U}}^{(m)} \mathfrak{D}_{\text{U} \rightarrow \text{A}}^{(m)} O - O \right\|_{j,m} \leq C_m \rho^j \|O\|_{j,m}. \quad (14.5)$$

This provides bidirectional equivalence modulo contracting and redundant operators without claiming a false microscopic bijection.

14.3 Why the proof is modular

The argument separates five logically distinct layers:

$$\boxed{\text{carrier} \longrightarrow \text{fingerprint} \longrightarrow \text{RG convergence} \longrightarrow \text{spectator quotient} \longrightarrow \text{external dictionary.}} \quad (14.6)$$

Carrier geometry. This layer supplies persistent spatial organization, a phase interval, and crossing and mark observables. It does not name a dynamic universality class.

Active fingerprint. The fingerprint records the symmetry representation, number of slow fields, conservation matrix, reversibility/FDT status, absorbing constraint, range, noise type, and tuning codimension.

RG dynamics. RG1–RG4 establish the common operational hierarchy, unique full-sequence attraction, linear spectrum, nonlinear scaling fields, dangerous-variable reconstruction, and

irrelevant-tail bounds.

Spectator adjudication. RG5 distinguishes removable gapped spectators from critical, hydrodynamic, protected, or relevantly coupled sectors. RG6 then assigns one noncolliding UMD subclass.

External equivalence. RG7 fixes the independently defined reference and constructs the scaling-field and observable dictionaries with quantitative reconstruction control.

The separation prevents two circularities: choosing a reference class after observing the exponent values, and deleting a sector because it is invisible to a preferred observable.

14.4 Relation to dynamic critical phenomena

The certified active sector has the standard Model-A fingerprint:

- one real scalar order parameter with $\mathbb{Z}_2$ symmetry;
- nonconserved relaxational dynamics;
- equilibrium noise and FDT;
- no absorbing-state condition;
- a short-range spatial kernel.

Above the upper critical dimension, the common leading fixed point is Gaussian:

$$\eta = 0, \quad \nu = \frac{1}{2}, \quad z = 2. \quad (14.7)$$

However, the quartic field remains registered with

$$y_u = 4 - d < 0. \quad (14.8)$$

It is spectrally irrelevant but dangerous for ordered, zero-mode, and finite-size observables. The theorem therefore does not confuse a Gaussian leading fixed point with a literally free microscopic model.

The result is narrower than the phenomenological Model-A family because it fixes $d > 4$, short range, equilibrium/FDT structure, weak coupling, and the PURE/QUOT condition. It is stronger than a symmetry-based universality conjecture because it provides the model-specific RG conjugacy, the observable dictionary, and a controlled reconstruction defect.

14.5 Interpretation of the filamentary phase

The filamentary carrier is physically and mathematically essential. It provides:

- the infinite-volume spatial substrate;
- a persistent phase interval;
- crossing-density and mark observables;

- stable geometric organization on which the active field is defined.

It does not, by itself, determine whether the field is conserved, whether FDT holds, whether a critical spectator survives, which fixed point attracts the flow, or whether the external class is Gaussian, interacting, absorbing, or nonequilibrium. Its precise role is therefore summarized by

$$\boxed{\text{filamentary phase} = \text{carrier-level universality datum}} \quad (14.9)$$

and

$$\boxed{\begin{aligned} \text{full universality class} &= \text{carrier} + \text{fingerprint} \\ &+ \text{RG signature} + \text{spectator data.} \end{aligned}} \quad (14.10)$$

This distinction explains both the positive restricted theorem and the failure of the unrestricted implication.

14.6 Spectators and the meaning of universality

Agreement on one observable cannot establish full equivalence. A decoupled massless field may leave the filamentary observable unchanged while adding a critical pole and enlarging the fixed-point spectrum. A conserved spectator may alter the dynamic class. A protected sector may change infrared data without changing local scalar bulk correlations.

The PURE/QUOT theorem therefore removes only sectors satisfying S1–S7. EXT sectors are not treated as defects; they are candidates for product or coupled universality subclasses. Universality here means insensitivity to microscopic details within a certified RG basin. It does not mean insensitivity to conservation laws, protected sectors, or relevant couplings.

14.7 Numerical–analytic interface

Computations, frozen selectors, and quantitative enclosures can legitimately:

- locate candidate branches;
- certify finite constants and remainder bounds;
- test noncancellation and stability hypotheses;
- expose counterexamples before a universal claim is made;
- supply reproducible gates inside an analytic implication.

They cannot infer an infinite-volume theorem from a finite fit, select the reference after seeing the outcome, replace full-sequence attraction with a long transient, or promote numerical exponent agreement to RG conjugacy. The outcome-blind freeze and first-failure ledger are therefore mathematical safeguards rather than merely reproducibility conventions.

14.8 Universal and nonuniversal data

The theorem identifies:

- the fixed point and its RG basin;
- relevant, marginal, dangerous, and irrelevant directions;

- universal exponent data;
- normalized scaling functions and amplitude ratios;
- the renormalized response/correlation dictionary;
- the quotient observable algebra.

It does not identify microscopic configurations or generators, bare couplings, absolute length and time units, individual field-normalization factors, or arbitrary finite-size data outside the scaling regime. Nonuniversal factors such as Z_O , Z_s , a_t , a_h , and a_u are calibration parameters, not failures of universality.

14.9 Limits and Atlas interpretation

The positive result excludes the marginal $d = 4$ branch, interacting $d < 4$ fixed points, conserved or mixed dynamics, absorbing-state classes, FDT-breaking nonequilibrium fixed points, long-range kernels outside the frozen basin, nonquotient EXT sectors, and sequences without unique full-sequence attraction. Accordingly,

$$\text{T5}_{\text{AG}, d>4}^+ \neq \text{T5}_{\text{unrestricted}}. \quad (14.11)$$

The former is proved; the latter is false.

The result is best viewed as the first certified chart of the future UMD Universality Atlas:

$$\mathcal{U}_{\text{UMD}}(M) = (\mathfrak{F}, \mathfrak{S}, \Sigma_{\text{RG}}, \text{Reference}, \text{Verdict}). \quad (14.12)$$

The present article supplies the signature format, spectator-indexed branch selector, proof-or-counterexample logic, first positive external entry, and the failure semantics required for later charts. Atlas completeness will mean that each admissible sequence receives a certified subclass or an explicit OUT/FAIL/UNRESOLVED verdict; it will not mean that every branch is forced into one class.

14.10 Conceptual lesson and discussion verdict

The word modular has a precise mathematical role. Carrier geometry can persist while the dynamics changes; the active fingerprint can remain fixed while a spectator changes; the spectrum can agree while nonlinear scaling fields differ; a fixed point can agree while the observable embedding fails; and an internal UMD subclass can exist without an external dictionary. The independent variation of these proof objects permits a strong positive theorem together with honest counterexample boundaries.

The controlled passage established here is

$\begin{aligned} &\text{UMD thermodynamic sequence} \longrightarrow \text{persistent filamentary phase} \\ &\longrightarrow \text{Gaussian Model-A UMD subclass} \longrightarrow \text{external Model-A RG equivalence.} \end{aligned}$	(14.13)
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Its significance lies equally in the positive equivalence and in the proof that carrier persistence, exponent matching, and a preferred observable are individually insufficient. This section adds no hypothesis and enlarges no theorem domain; it interprets the exact result of Theorem 12.1.

15 Conclusion

The problem addressed in this article was whether persistence of an asymptotic filamentary phase in Universal Modular Dynamics can be promoted from an infinite-volume geometric statement to a full renormalization-group universality-class equivalence with an independently defined external critical theory. The answer has two complementary parts. The unrestricted implication is false, whereas a precisely delimited positive theorem holds on the scalar Gaussian Model-A branch above its upper critical dimension.

The positive result is

$$\boxed{\text{T5}_{\text{A}_G, d>4}^+ : \quad Q_{\text{sp}} M \simeq_{\text{RG}} \text{A}_{\text{ref}}(d), \quad d > 4.} \quad (15.1)$$

Here M is an admissible critical UMD thermodynamic sequence in the frozen domain $\mathfrak{D}_{\text{T5}}^+(d)$, Q_{sp} is the canonical quotient by an S1–S7-removable spectator sector, and $\text{A}_{\text{ref}}(d)$ is the outcome-blind external scalar Gaussian Model-A reference. Equivalently, the theorem applies when the persistent carrier is supplemented by the scalar $\mathbb{Z}_2$ -odd, nonconserved, short-range, reversible/FDT fingerprint, codimension-one critical tuning, the complete RG1–RG7 certificates, and the PURE/QUOT spectator condition.

The equivalence in (15.1) is stronger than agreement of a few critical exponents. It comprises unique full-sequence attraction to the common Gaussian Ornstein–Uhlenbeck fixed point, equality of the normalized linearized spectra and multiplicities, conjugacy of the nonlinear scaling fields, preservation of the dangerously irrelevant quartic coordinate, a bidirectional scaling-field and observable dictionary, equality of normalized RG signatures, and scale-decaying reconstruction error on the quotient observable algebra. In particular,

$$\eta = 0, \quad \nu = \frac{1}{2}, \quad z = 2, \quad y_u = 4 - d < 0, \quad (15.2)$$

with the indispensable dangerous-irrelevance qualification for zero-mode, ordered-phase, and finite-size thermodynamic observables. Microscopic generators, metric factors, bare couplings, and configurations need not coincide; RG universality is an equivalence of infrared signatures and observable reconstructions, not microscopic identity.

The proof is noncircular because each logical transition is carried by a separate certificate:

$$\boxed{\text{carrier} \longrightarrow \text{fingerprint} \longrightarrow \text{RG1} - \text{RG4} \longrightarrow \text{RG5} \longrightarrow \text{RG6} \longrightarrow \text{RG7}.} \quad (15.3)$$

Carrier persistence supplies the spatial substrate but does not determine the dynamic class. The fingerprint fixes the candidate branch but does not prove basin entry. RG1–RG4 establish the common hierarchy, attraction, spectrum, nonlinear scaling fields, and irrelevant-tail control. RG5 adjudicates spectators, RG6 assigns a noncolliding internal UMD subclass, and RG7 constructs the external equivalence. No reference class is selected after observing exponents, and no sector is discarded merely because it is invisible to a preferred observable.

The exact claim boundary is therefore

$$\boxed{\begin{array}{c} \text{persistent asymptotic UMD filamentary phase} \not\Rightarrow \\ \text{full external universality-class equivalence} \end{array}} \quad (15.4)$$

Conservation laws, absorbing constraints, long-range interactions, nonequilibrium noise, non-

Gaussian or multicritical attraction, and critical, hydrodynamic, protected, or relevantly coupled spectator sectors can all change the infrared class without destroying the carrier. The present theorem consequently excludes $d = 4$ and $d < 4$, non-Model-A fingerprints, uncontrolled EXT sectors, and sequences lacking unique full-sequence RG convergence. Thus

$$\boxed{\text{T5}_{\text{AG},d>4}^+ = \text{PROVED}, \quad \text{T5}_{\text{unrestricted}} = \text{FALSE}.} \quad (15.5)$$

These verdicts do not conflict: the negative result prevents overclaiming, while the positive result identifies the complete subclass certified by the proof objects of this article.

The work also fixes the interface for the next programme. The UMD Universality Atlas is to implement the map

$$\boxed{\mathcal{U}_{\text{UMD}} : \{\text{UMD critical sequences}\} \longrightarrow \{\text{fingerprint- and spectator-indexed external classes}\}} \quad (15.6)$$

by assigning each admissible sequence a certified external subclass or an explicit **OUT**, **FAIL**, or **UNRESOLVED** verdict. The present article contributes the signature format, spectator registry, quotient rules, deterministic first-failure semantics, branch-noncollision principle, and the first complete positive Atlas entry, $\text{T5}_{\text{AG},d>4}^+$. Extensions to the marginal and interacting dimensions, conserved and mode-coupled dynamics, absorbing and nonequilibrium branches, and controlled EXT sectors require new certificates rather than a relaxation of the present theorem standard.

The final takeaway is therefore precise. A persistent filamentary phase is an essential carrier-level universality datum, but it is not a complete universality classifier. Once the active-field fingerprint, RG1–RG7, and the **PURE**/**QUOT** spectator condition are certified, the $d > 4$ quartic modular UMD branch is RG-equivalent to the external Gaussian Model-A class. This closes the T5 proof-or-counterexample programme for its frozen domain and replaces a broad conjectural implication with one proved theorem, one exact counterexample boundary, and a rigorous route toward a classified UMD universality atlas.

A Complete Assumption-to-Theorem Dependency Ledger

A.1 Purpose and scope

This appendix records the complete dependency structure of the restricted positive theorem. It identifies the direct use of every assumption, the proof object certifying each RG gate, the permitted claim tier, and the first-failure verdict produced when a contract is absent. It introduces no new hypothesis and does not enlarge the domain $\mathfrak{D}_{T5}^+(d)$ defined in Section 12.1.

The object traced below is

$$T5_{A_G, d>4}^+ : \quad Q_{\text{sp}} M \simeq_{\text{RG}} A_{\text{ref}}(d), \quad M \in \bigcap_{i=1}^{15} \mathfrak{D}(H_i). \quad (\text{A.1})$$

A.2 Complete assumption-to-result ledger

Table 8: Assumptions, their direct outputs, and first-failure codes.

ID	Exact content	Direct output	Primary location; failure
H_1	Admissible finite UMD sequence, thermodynamic embedding, uniform quasi-locality, and microscopic consistency.	Well-defined infinite-volume comparison object.	Section 3; F_0 .
H_2	Outcome-blind persistent filamentary carrier satisfying the T1–T4 observable contracts.	Carrier phase and its physical observables.	Prior carrier theorem; F_1 .
H_3	Stabilization, local-weak transfer, uniform block contiguity, and dependence-defect control.	Transfer of carrier observables to the limit.	Prior carrier theorem; F_1 .
H_4	One real $\mathbb{Z}_2$ -odd scalar slow mode.	Scalar candidate branch.	Section 3; F_2 .
H_5	Nonconserved, purely relaxational active field.	Model-A dynamic fingerprint.	Section 3; F_2 .
H_6	Detailed balance and FDT; no absorbing-state constraint.	Equilibrium response/noise structure.	Section 3; F_2 .
H_7	Short-range active kernel and critical tuning codimension $r = 1$.	Frozen comparison branch and critical surface.	Section 3; F_2 .
H_8	$d > 4$ (integer $d \geq 5$ in the lattice realization).	Gaussian basin with $y_u = 4 - d < 0$.	Section 5; F_2 .
H_9	RG1 common outcome-blind coarse graining, Banach topology, and relevant-field chart.	Common RG comparison category.	Section 4; F_3 .

ID	Exact content	Direct output	Primary location; failure
H_{10}	RG2 unique full-sequence attraction to the Gaussian Ornstein–Uhlenbeck fixed point.	Basin entry and fixed-point convergence.	Section 5; F_4 .
H_{11}	RG3 equality of relevant, marginal, dangerous, and irrelevant spectra with multiplicities.	Linearized spectrum equality.	Section 6; F_5 .
H_{12}	RG4 nonlinear conjugacy, dangerous-quartic reconstruction, dynamic locality, and uniform irrelevant-tail bounds.	Common nonlinear normal form.	Sections 7 and 8; F_5 .
H_{13}	RG5 PURE sector, or QUOT sector satisfying S1–S7.	Canonical spectator quotient; EXT rejection.	Section 9; F_6 .
H_{14}	RG6 unique fingerprint- and spectator-indexed subclass assignment.	Membership in $A_G^{\text{PURE/QUOT}}(d)$.	Section 10; F_7 .
H_{15}	RG7 external scaling-field and observable dictionaries, signature equality, and reconstruction estimate.	External Gaussian Model-A equivalence.	Section 11; F_8 .

Every contract in Table 8 has a direct downstream use. None is decorative. Conversely, every component of (A.1) has an upstream contract.

A.3 RG1–RG7 proof-object traceability

Table 9: RG certificates and their consumers.

Gate	Required proof object	Certified consequence	Consumed by
RG1	Common outcome-blind block semigroup, embeddings into one Banach hierarchy, and relevant-field chart.	UMD and reference trajectories are comparable in one category.	RG2–RG4 and RG7.
RG2	Tightness, asymptotic compactness, unique full-sequence attraction, and exact Gaussian modular subclass.	Common Gaussian OU fixed point in $d > 4$.	RG3, RG4, and RG6.
RG3	UMD/reference linearized operators, invariant subspaces, eigenvalues, and algebraic multiplicities.	Equality of the complete normalized linear spectrum.	RG4, RG6, and RG7.

Gate	Required proof object	Certified consequence	Consumed by
RG4	Nonlinear scaling-field conjugacy, quartic dangerous-variable ledger, polymer/memory contraction, and uniform truncation bounds.	Common nonlinear normal form and controlled finite-size reconstruction.	RG6 and RG7.
RG5	Spectator registry, S1–S7 quotient theorem, and nondecoupling counterexample gate.	Preservation of PURE/QUOT data and exclusion of uncontrolled EXT sectors.	RG6 and RG7.
RG6	Exhaustive, noncolliding fingerprint- and spectator-indexed selector.	Unique internal UMD universality subclass.	RG7 and Theorem 12.1.
RG7	Independently frozen external reference, bidirectional dictionaries, normalized-signature equality, and reconstruction bound.	External Gaussian Model- A RG equivalence.	Theorem 12.1.

The technically extended RG4 branch factors as

$$\begin{aligned}
& \text{Gaussian modular parent} \longrightarrow \text{linear spectrum} \longrightarrow \text{quartic parent} \\
& \longrightarrow \text{static } d > 4 \text{ polymer flow} \longrightarrow \text{critical stable trajectory} \\
& \longrightarrow \text{memory/FDT state space} \longrightarrow \text{dynamic locality} \\
& \longrightarrow \text{shell-semigroup propagation} \longrightarrow \text{RG4}.
\end{aligned} \tag{A.2}$$

In particular, no dynamic conclusion is imported solely from the static ϕ^4 construction.

A.4 Claim-tier map

The theorem architecture admits four strictly ordered claim tiers:

$$\begin{aligned}
& \text{Tier I} = \text{Carrier}(H_1, H_2, H_3), \\
& \text{Tier II} = \text{Fingerprint}(H_4, \dots, H_8), \\
& \text{Tier III} = \text{InternalClass}(\text{RG1}, \dots, \text{RG6}), \\
& \text{Tier IV} = \text{ExternalEquivalence}(\text{RG7}).
\end{aligned} \tag{A.3}$$

Table 10: Permitted claims at each tier.

Tier	Licensed claim	Required input	Not licensed
I	Persistent infinite-volume filamentary phase and T1–T4 observables.	H_1 – H_3 .	Active fingerprint or external class.
II	Scalar, nonconserved, equilibrium, short-range candidate branch in $d > 4$.	H_4 – H_8 plus Tier I.	Fixed-point attraction.
III	Unique $A_G^{\text{PURE/QUOT}}$ internal UMD subclass.	RG1–RG6 plus Tiers I–II.	External equivalence without RG7.
IV	RG equivalence with the frozen external Gaussian Model-A reference.	RG7 plus Tiers I–III.	Microscopic identity.

Consequently,

$$\boxed{\text{Tier I} \not\Rightarrow \text{Tier IV.}} \quad (\text{A.4})$$

Tier IV entails Tier III inside the frozen comparison category, but carrier persistence alone entails neither the Model-A fingerprint nor external equivalence.

A.5 Failure-alternative crosswalk

Table 11: First-failure semantics and allowed continuations.

Code	First failed input	Surviving claim	Forbidden claim; valid continuation
F ₀	H_1	Finite UMD definitions.	No thermodynamic theorem; repair the admissible embedding.
F ₁	H_2 – H_3	Admissible UMD thermodynamic sequence.	No persistent-carrier claim; prove or refute carrier transfer.
F ₂	H_4 – H_8	Carrier may persist.	No Model-A branch; select the correct fingerprint branch.
F ₃	RG1/ H_9	Carrier and fingerprint.	No common RG comparison; construct an operational coarse graining.
F ₄	RG2/ H_{10}	Common RG category.	No unique fixed-point class; classify multiple, cyclic, or runaway limits.
F ₅	RG3–RG4/ H_{11} – H_{12}	Candidate fixed point.	No full-signature equality; construct a new spectral/nonlinear subclass.
F ₆	RG5/ H_{13}	Active-field branch.	No PURE/QUOT theorem; open a product or coupled EXT programme.
F ₇	RG6/ H_{14}	Partial RG signature.	No unique internal class; refine or split the classifier.

Code	First failed input	Surviving claim	Forbidden claim; valid continuation
F_8	$RG7/H_{15}$	Certified internal UMD subclass.	No external equivalence; construct the missing reference dictionary.
P	None	Tiers I–IV and $T_{AG,d>4}^+$.	No microscopic-identity or unrestricted-universality claim.

The ordered first-failure rule is deterministic: adjudication stops at the first false or unresolved contract. A failure code denies only the claim that depends on the missing object; it does not assert absence of all criticality or all universality.

A.6 Counterexample-to-contract map

Table 12: Mechanisms preventing the unrestricted implication.

Mechanism	Failed contract	Adjudicated branch
Conserved active order parameter	H_5	Model-B-type candidate.
Active scalar coupled to a conserved density	H_5 or H_{13}	Model-C/coupled candidate.
Massless spectator	$H_{13}/S1$ – $S2$	EXT product fixed point.
Protected topological or anomalous sector	$H_{13}/S5$	EXT protected branch.
Relevant active–spectator coupling	$H_{13}/S3$	EXT coupled branch.
Relevant FDT-breaking noise or memory	H_6 or H_{12}	Nonequilibrium branch.
Absorbing-state constraint	H_6	Absorbing-state branch.
Relevant long-range kernel	H_7	Long-range branch.
$d = 4$ marginal quartic flow	H_8	Logarithmic marginal branch.
$d < 4$ interacting fixed point	H_8 or H_{10}	Wilson–Fisher-type branch.
Subsequence-dependent RG limits	H_{10}	FAIL/UNRESOLVED.

A.7 Circularity firewall and minimality

The dependency graph enforces the following firewall:

- (i) carrier extraction is frozen before phase outcomes;
- (ii) the active fingerprint is fixed before external-class adjudication;
- (iii) the reference is not selected from measured exponent values;
- (iv) spectator quotienting requires $S1$ – $S7$, not mere invisibility to one observable;
- (v) exponents follow from signature equality rather than define it;
- (vi) the dangerous quartic coordinate is retained before singular and finite-size observables are reconstructed;
- (vii) reverse reconstruction is asserted only on the RG quotient observable algebra.

The assumptions are minimal relative to this proof architecture. Removing H_1 destroys the thermodynamic object; removing H_2 – H_3 destroys the filamentary premise; changing H_4 – H_8 permits a different dynamic branch; removing any of RG1–RG4 deletes a component of signature equality; removing RG5 permits a spectator counterexample; removing RG6 destroys unique internal classification; and removing RG7 leaves no certified external equivalence. This is proof-relative minimality: future arguments may replace a contract, but the present proof cannot simply omit it.

A.8 Traceability verdict

The audit identifies no orphan assumption, no theorem claim without an upstream proof object, no backward dependency from observed exponents to branch selection, no spectator quotient without S1–S7, and no external-equivalence claim before RG7. Therefore

$$\boxed{\text{Appendix A dependency and traceability audit: PASS.}} \quad (\text{A.5})$$

This verdict preserves both the positive theorem Theorem 12.1 and its exact failure boundary; it changes neither the theorem domain nor the article-level claim hierarchy.

B Exact UMD/Reference Semantics and Reproducibility Contracts

B.1 Exactness convention

This appendix fixes the state spaces, maps, quantifiers, constants, and reproducibility objects used by RG1–RG7. Here *exact* means that an object and its permissible use are specified without ambiguity. It does not assign an artificial numerical value to a theorem constant that is existential or to a calibration factor that is nonuniversal. We keep four categories separate:

- (i) universal exact data;
- (ii) chosen reference normalizations;
- (iii) nonuniversal metric factors;
- (iv) quantified control constants.

B.2 UMD and reference RG state spaces

On the frozen scalar critical branch, a local UMD RG state is represented by

$$\begin{aligned} X_U = (t_U, h_U, u_U, \Gamma_U, Z_{\phi,U}, Z_{\tilde{\phi},U}, \\ W_U^{\text{loc}}, W_U^{\text{mem}}, S_U) \in \mathcal{X}_U. \end{aligned} \quad (\text{B.1})$$

The first three coordinates are the thermal, magnetic, and quartic scaling fields; Γ_U is the kinetic coefficient; $Z_{\phi,U}$ and $Z_{\tilde{\phi},U}$ normalize the field and response field; W_U^{loc} and W_U^{mem} are the local-polymer and FDT-paired memory/noise remainders; and S_U is the complete spectator registry.

The external state space $\mathcal{X}_A$ is generated by the independently defined Model-A object of Section 11.1. Its state is

$$X_A = (t_A, h_A, u_A, \Gamma_A, Z_{\phi, A}, Z_{\tilde{\phi}, A}, W_A^{\text{loc}}, W_A^{\text{mem}}). \quad (\text{B.2})$$

After the frozen spatial, temperature, and time normalization, its Gaussian fixed representative has

$$t_A = h_A = u_A = 0, \quad T = \Gamma_A = 1. \quad (\text{B.3})$$

The coordinate u_A remains present in the enlarged chart even at this fixed point because it is dangerously irrelevant.

Both theories map into the common scaling space

$$\mathcal{Y} = E_t \oplus E_h \oplus E_u \oplus E_\Gamma \oplus E_Z \oplus E_{\text{loc}} \oplus E_{\text{mem}}, \quad (\text{B.4})$$

with coordinate $y = (t, h, u, \Gamma, Z_\phi, Z_{\tilde{\phi}}, \mathbf{w}, \mathbf{m})$ and norm

$$\begin{aligned} \|y\|_{\mathcal{Y}} = & \alpha_t |t| + \alpha_h |h| + \alpha_u |u| + \alpha_\Gamma |\Gamma - 1| \\ & + \alpha_Z (|Z_\phi - 1| + |Z_{\tilde{\phi}} - 1|) + \|\mathbf{w}\|_{\mathcal{P}} + \|\mathbf{m}\|_{\mathcal{M}}. \end{aligned} \quad (\text{B.5})$$

All weights are positive and frozen before comparison. They specify an equivalent local norm, not universal amplitudes.

B.3 Common RG maps and conjugacy semantics

Let $\mathcal{R}_U$ denote the one-shell RG transformation on the spectator quotient $\mathcal{X}_U^q$, and let $\mathcal{R}_A$ denote its reference counterpart. For fixed block factor $L > 1$, their common normal form is

$$\mathcal{N}_L(y) = (L^2 t, L^{(d+2)/2} h, L^{4-d} u, \Gamma, Z_\phi, Z_{\tilde{\phi}}, K_L \mathbf{w}, K_L^m \mathbf{m}) + \mathcal{Q}(y), \quad (\text{B.6})$$

where

$$\mathcal{Q}(0) = D\mathcal{Q}(0) = 0, \quad \max\{\|K_L\|, \|K_L^m\|\} \leq \rho < 1. \quad (\text{B.7})$$

The local charts Ψ_U and Ψ_A of Equation (11.5) define

$$\mathfrak{C} = \Psi_A^{-1} \circ \Psi_U, \quad \mathfrak{C}^{-1} = \Psi_U^{-1} \circ \Psi_A, \quad \mathfrak{C}\mathcal{R}_U = \mathcal{R}_A\mathfrak{C}. \quad (\text{B.8})$$

This is a local analytic bijection of RG quotient charts. It is not a bijection of microscopic configurations.

B.4 Bidirectional scaling-field dictionary

The forward dictionary has the local expansion

$$\begin{aligned} t_A &= a_t t_U + O(\|y_U\|^2), & h_A &= a_h h_U + O(\|y_U\|^2), \\ u_A &= a_u u_U + O(\|y_U\|^2), & \Gamma_A &= a_\Gamma \Gamma_U + O(\|y_U\|^2), \\ \mathbf{w}_A &= B_w \mathbf{w}_U + O(\|y_U\|^2), & \mathbf{m}_A &= B_m \mathbf{m}_U + O(\|y_U\|^2). \end{aligned} \quad (\text{B.9})$$

Here $a_t a_h a_u a_\Gamma \neq 0$, while B_w and B_m are boundedly invertible on the retained irrelevant charts. The reverse dictionary is the local analytic inverse. These coefficients are nonuniversal calibration

data; their invertibility, rather than equality to a preferred number, enters the theorem.

The dictionary preserves the relevant, dangerous, and irrelevant decomposition:

$$\mathfrak{C}(E_U^{\text{rel}}) = E_A^{\text{rel}}, \quad \mathfrak{C}(E_U^{\text{dang}}) = E_A^{\text{dang}}, \quad \mathfrak{C}(E_U^{\text{irr}}) = E_A^{\text{irr}}. \quad (\text{B.10})$$

B.5 Observable dictionary and reconstruction quantifiers

Let O_U and $\tilde{O}_U$ be the normalized UMD slow field and response field. With $Z_O, Z_s > 0$,

$$\phi_A = Z_O^{-1/2} O_U, \quad \tilde{\phi}_A = Z_O^{1/2} Z_s^{-1} \tilde{O}_U, \quad s_A = Z_s s_U. \quad (\text{B.11})$$

Consequently,

$$\begin{aligned} C_A(x, s) &= Z_O^{-1} C_U(x, Z_s^{-1} s), & R_A(x, s) &= Z_s^{-1} R_U(x, Z_s^{-1} s), \\ \chi_A &= Z_O^{-1} \chi_U, & S_A(k, \omega) &= Z_O^{-1} Z_s S_U(k, Z_s \omega). \end{aligned} \quad (\text{B.12})$$

Composite fields are transported by an invertible mixing matrix within each fixed symmetry and scaling sector; inequivalent symmetry sectors are not mixed.

For every finite observable degree m and compact scaled momentum–frequency window K , there exist

$$C_{m,K} < \infty, \quad j_0(m, K) < \infty, \quad 0 < \rho < 1, \quad (\text{B.13})$$

such that, for all $j \geq j_0(m, K)$ and all observables in the stated quotient algebra,

$$\left\| \mathfrak{D}_{A \rightarrow U}^{(m)} \mathfrak{D}_{U \rightarrow A}^{(m)} O - O \right\|_{j,m;K} \leq C_{m,K} \rho^j \|O\|_{j,m;K}, \quad (\text{B.14})$$

with the symmetric $A \rightarrow U \rightarrow A$ estimate. The quantifier order is

$$\forall m, K \exists C_{m,K}, j_0, \rho \forall j \geq j_0 \forall O. \quad (\text{B.15})$$

No uniformity in unbounded composite degree, all frequencies, or an unspecified singular $u = 0$ zero-mode limit is claimed.

B.6 Dangerous-irrelevance prescription

For $d > 4$,

$$y_u = 4 - d < 0, \quad u_j = L^{(4-d)j} u_0 + O(u_0^2). \quad (\text{B.16})$$

For regular local observables, $u_j \rightarrow 0$ may be taken after establishing the chart. For zero-mode, ordered-phase, and designated finite-size observables, the valid order is: retain u_j , evaluate the observable in the blown-up coordinates, take the prescribed joint scaling limit, and only then simplify the asymptotic u_j -dependence. Setting $u = 0$ before this reconstruction is outside the theorem.

B.7 Spectator-quotient contract

The relation $X_U \sim_{\text{sp}} X'_U$ holds exactly when the representatives have the same active coordinates and differ only by spectator components satisfying all seven contracts of Theorem 9.1:

S1 no additional critical pole;

- S2** a uniform static gap and dynamic mixing bound;
- S3** strict RG irrelevance of every generated active–spectator coupling;
- S4** local Schur-complement closure in the RG4 stable ideal;
- S5** no anomaly, protected topological mode, or retained superselection datum;
- S6** no independent overlap with the retained critical observable algebra;
- S7** analytic determinant contribution and controlled reconstruction defect.

The resulting space is

$$\mathcal{X}_U^q = \mathcal{X}_U / \sim_{\text{sp}}, \quad Q_{\text{sp}} \mathcal{R}_U = \overline{\mathcal{R}}_U Q_{\text{sp}}. \quad (\text{B.17})$$

Failure of any S-contract yields EXT or UNRESOLVED, never QUOT. Thus no critical or protected sector is silently identified with zero.

B.8 Constants ledger

Table 13: Universal exact data in the certified branch.

Symbol	Exact value	Role
d	$d > 4$	theorem domain
r	1	tuning codimension
y_t	2	thermal eigenvalue
y_h	$(d + 2)/2$	magnetic eigenvalue
y_u	$4 - d < 0$	dangerous quartic eigenvalue
η	0	anomalous field exponent
ν	$1/2$	correlation-length exponent
z	2	dynamic exponent

The coefficient of $|\nabla\phi|^2/2$, the reference temperature T , the fixed-point kinetic coefficient Γ_* , and the fixed-point field normalizations are chosen to equal one. These are coordinate choices. The factors

$$a_t, a_h, a_u, a_\Gamma, Z_O, Z_s > 0 \quad (\text{B.18})$$

are nonuniversal after orientation is fixed. Finally, $L > 1$, $\rho_{\text{loc}}, \rho_{\text{mem}} \in (0, 1)$, finite constants $C, C_{m,K}$, finite onset scales $j_0(m, K)$, positive chart radii, and positive norm weights are quantified control data. No unstated numerical value is assigned to them.

B.9 Reproducibility protocol and frozen manifest

A reproduction of the theorem must: identify the exact H_1 – H_{15} domain; verify the fingerprint before reference selection; instantiate the common RG map and chart; retain u ; run S1–S7 in order; verify RG1–RG7 in dependency order; compare normalized signatures; apply the observable dictionary only in its quantified norm and window; return the first failure code when necessary; and preserve the resulting claim tier. Finite-size exponent fitting alone is not a reproduction of the theorem.

Table 14: Frozen article-module manifest for the theorem and closure layer.

File	SHA-256
UMD_T5_ARTICLE_M5_section4_RG1. tex	1bca38360c73f8948c196f7c73b2492366d081cf852510d2 7586083e715d94ec
UMD_T5_ARTICLE_M6_section5_RG2. tex	f41b73aedd1bf4e45741b80f5950dbfe36b7b85c13236082 70df406842308979
UMD_T5_ARTICLE_M7_section6_RG3. tex	16969288cd9612ffb2bd5220f42fc06df032dae3f66053a5 126d16a90b15c49e
UMD_T5_ARTICLE_M8_section7_RG4_ core.tex	019ab171fcbc89716189656c6d3e401e1145ca5bbce7bcbd e09fa585a7ae1fc5
UMD_T5_ARTICLE_M9_section8_RG4_ final.tex	0c2355114e50743d46c660379300185619a9bdec71d1c8eb 99b37ea0ebb4dd0e
UMD_T5_ARTICLE_M10_section9_RG5. tex	9184006cce6d187fc4f633b973482a4dc3ab0edc26583416 7cab70d491c7c315
UMD_T5_ARTICLE_M11_section10_RG6. tex	168face362fe443f45c1cc5643c72146d21d2f56195c496e 816414606441b061
UMD_T5_ARTICLE_M12_section11_RG7. tex	066e0615d826019683db0539300d36de88f498fa7faf3b90 d1b46acd03b26da5
UMD_T5_ARTICLE_M13_section12_ final_T5.tex	a592139d5dff4efd3bf66dad722bda7e8c48886fee48d759 8694a85611800c24
UMD_T5_ARTICLE_M14_outlook.tex	b759860f6bf6719cecb08273227b259406323fb66f0af12f fc742a9aad0981dd
UMD_T5_ARTICLE_M15_discussion.tex	a9bd2900ff48d522bc13334d05d56b30ae6a6ea14c0276f3 684cb7001869fcd6
UMD_T5_ARTICLE_M16_conclusion.tex	9610cc6517c8d7ce313c64d26bf259926628c5267221b11a d597070b36178f98
UMD_T5_ARTICLE_M17_appendixA.tex	f530dce26bfcd2d1cd5f0cab3851d965b2777f082a31f9a8 4c361ecaec4611a0

The checksum of this Appendix B file is release metadata and must be recorded after the file is frozen; embedding it here would change the hashed object. Earlier proof stages remain identified by their exact stage names and frozen semantic contracts. A hash not independently retained is never reconstructed from memory.

B.10 Semantic and reproducibility verdict

The registry fixes the meaning and quantified scope of every state-space component, conjugacy, dictionary, quotient, and constant category used by the final theorem. It claims neither a microscopic bijection nor numerical universality of calibration factors, and it provides no extension to $d \leq 4$ or to an uncontrolled EXT branch. Therefore

$$\boxed{\text{Appendix B semantic and reproducibility audit: PASS.}} \quad (\text{B.19})$$

C Formal RG-Universality Relation and Universality-Atlas Interface

C.1 Purpose and comparison objects

This appendix formalizes RG-universality equivalence, proves that it is an equivalence relation on the certified comparison domain, records counterexamples to weaker selectors, and freezes the interface inherited by the future UMD Universality Atlas. It does not enlarge $\mathfrak{D}_{T_5}^+(d)$.

An RG theory object is the tuple

$$\mathbf{T} = (\mathcal{X}^q, \mathcal{R}, \mathcal{F}_*, \mathfrak{F}, \mathfrak{S}, \mathcal{A}, \mathcal{D}), \quad (\text{C.1})$$

where $\mathcal{X}^q$ is the local state space after admissible redundant and spectator quotienting; $\mathcal{R}$ is the one-shell RG map; $\mathcal{F}_*$ is the selected full-sequence attractor; $\mathfrak{F}$ is the active-field fingerprint; $\mathfrak{S}$ is the spectator index; $\mathcal{A}$ is the renormalized observable/response algebra; and $\mathcal{D}$ is the dangerous-variable ledger. Its normalized signature is

$$\Sigma_{\text{RG}}(\mathbf{T}) = [\mathfrak{F}, \mathfrak{S}, \mathcal{F}_*, D\mathcal{R}|_{\mathcal{F}_*}, \mathcal{N}_*, \mathcal{D}, \mathcal{O}, r]_{\cong}. \quad (\text{C.2})$$

C.2 Admissible equivalence morphisms

Let $\mathbf{T}_1$ and $\mathbf{T}_2$ have the same fixed spatial dimension. An admissible morphism $\mathfrak{C}_{12} : \mathbf{T}_1 \rightsquigarrow \mathbf{T}_2$ consists of:

- (i) neighborhoods $\mathcal{N}_i \subset \mathcal{X}_i^q$;
- (ii) a local analytic bijection $\mathfrak{C}_{12} : \mathcal{N}_1 \rightarrow \mathcal{N}_2$;
- (iii) preservation of relevant, marginal, dangerous, and irrelevant subspaces;
- (iv) RG intertwining with contracting defect,

$$\left\| \mathfrak{C}_{12} \mathcal{R}_1^j x - \mathcal{R}_2^j \mathfrak{C}_{12} x \right\| \leq C_{12} \rho_{12}^j \|w(x)\|, \quad 0 < \rho_{12} < 1; \quad (\text{C.3})$$

- (v) an invertible scaling-field dictionary;
- (vi) an observable-algebra isomorphism $\mathfrak{D}_{12} : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ preserving symmetry, causality, allowed operator mixing, and $\mathcal{D}$;
- (vii) bidirectional quotient reconstruction with a scale-decaying defect.

Metric factors may differ. Dimension, nonredundant spectral multiplicities, protected data, and dangerous-variable semantics may not.

Definition C.1 (RG-universality equivalence). Write

$$\mathbf{T}_1 \simeq_{\text{RG}} \mathbf{T}_2 \quad (\text{C.4})$$

if an admissible morphism $\mathfrak{C}_{12}$ and its admissible local inverse exist and

$$\Sigma_{\text{RG}}(\mathbf{T}_1) = \Sigma_{\text{RG}}(\mathbf{T}_2). \quad (\text{C.5})$$

This definition is stronger than equality of finitely many exponents and weaker than microscopic isomorphism.

C.3 Equivalence-relation theorem

Theorem C.2 (RG equivalence relation). *On the RG1–RG7-certified comparison domain, $\simeq_{\text{RG}}$ is reflexive, symmetric, and transitive.*

Proof. Reflexivity follows from the identity state-space chart and identity observable map, for which both defects vanish. For symmetry, an admissible $\mathfrak{C}_{12}$ has an analytic inverse on a possibly smaller neighborhood; local Lipschitz control converts (C.3) into a reverse bound $C_{21}\rho_{21}^j$, with $0 < \rho_{21} < 1$, and signature equality is symmetric.

For transitivity, suppose $\mathbf{T}_1 \simeq_{\text{RG}} \mathbf{T}_2$ and $\mathbf{T}_2 \simeq_{\text{RG}} \mathbf{T}_3$. After restricting to compatible neighborhoods, define

$$\mathfrak{C}_{13} = \mathfrak{C}_{23} \circ \mathfrak{C}_{12}, \quad \mathfrak{D}_{13} = \mathfrak{D}_{23} \circ \mathfrak{D}_{12}. \quad (\text{C.6})$$

The two defects compose into a bound $C_{13}\rho_{13}^j$, where $\rho_{13} = \max\{\rho_{12}, \rho_{23}\} < 1$, after local Lipschitz constants are absorbed into C_{13} . The scaling dictionaries and observable isomorphisms compose, while signature equality is transitive. $\square$

The resulting universality class is the quotient set

$$[\mathbf{T}]_{\text{RG}} := \{\mathbf{T}' : \mathbf{T}' \simeq_{\text{RG}} \mathbf{T}\}. \quad (\text{C.7})$$

The positive theorem therefore states

$$[Q_{\text{sp}}M]_{\text{RG}} = [\mathbf{A}_{\text{ref}}(d)]_{\text{RG}}, \quad M \in \mathfrak{D}_{\text{T5}}^+(d). \quad (\text{C.8})$$

C.4 Counterexample library

Table 15: Counterexamples to incomplete universality selectors.

ID	Construction	Possible agreement	Decisive mismatch and verdict
CE1	Persistent carrier with a conserved scalar field.	Carrier geometry and static symmetry.	k^4 rather than k^2 soft pole; F_2 , Model-B candidate.
CE2	Model-A active field plus a decoupled massless scalar.	Filament observable and active exponents.	Additional critical spectrum; F_6 , EXT product.
CE3	Active field plus a relevantly coupled critical spectator.	Carrier and microscopic symmetry.	Joint beta functions and fixed point; F_6 , EXT coupled.
CE4	Active scalar plus a protected topological sector.	All local scalar correlators.	Protected or anomalous invariant; F_6 , EXT protected.
CE5	FDT-breaking coloured noise.	Selected static exponents.	Response/correlation relation and entropy production; F_2/F_5 , nonequilibrium.

ID	Construction	Possible agreement	Decisive mismatch and verdict
CE6	Scalar quartic branch in $d = 4$.	Mean-field powers.	Universal logarithmic corrections; outside H_8 , marginal branch.
CE7	Scalar quartic branch in $d < 4$.	Symmetry and conservation fingerprint.	Interacting fixed point and anomalous spectrum; outside H_8/H_{10} , Wilson–Fisher branch.
CE8	RG flow with two sequential limits.	Long finite crossover.	No unique full-sequence attraction; F_4 .
CE9	Matching fixed-point spectrum with noninvertible observable embedding.	Critical exponents.	Failed reconstruction algebra; F_8 .
CE10	Long-range kernel tuned to mimic one exponent.	One measured exponent.	Kernel eigenstructure and scaling functions; F_2/F_5 , long-range branch.

Each construction violates at least one explicit contract and therefore does not contradict Theorem 12.1. It follows in particular that same carrier geometry, symmetry alone, finitely many equal exponents, one matching correlator, or a fixed point without an observable dictionary is not a sufficient universality selector.

C.5 Nonquotient EXT-sector registry

An inadmissible spectator is retained and assigned one of the following provisional branches.

EXT-P: product-critical spectator. The candidate fixed point is

$$\mathcal{F}_* = \mathcal{F}_*^A \times \mathcal{F}_*^S. \quad (\text{C.9})$$

Both factors must have unique full-sequence attraction, all cross-couplings must be strictly irrelevant, the full spectrum must be their multiset union, and the observable algebra must be a controlled tensor product. Its reference is the product class, not Model A alone.

EXT-C: coupled critical spectator. A joint state space must retain $(\beta_A, \beta_S, \beta_{AS})$, exhibit a stable joint attractor, certify its complete spectrum and tuning codimension, and construct a joint external reference and dictionary.

EXT-H: hydrodynamic or conserved spectator. Continuity equations, conserved noise, hydrodynamic poles, and momentum-sector coercivity must be explicit. The correct Model-B, Model-C, or higher reference must be proved; $z = 2$ cannot be inherited from the present Model-A entry.

EXT-T: protected, topological, or anomalous spectator. Protected labels and anomaly data remain in the signature. Local correlators and protected invariants must both be compared,

and quotienting is forbidden unless the protected data are proved trivial.

EXT-R: runaway or nonstationary branch. This branch includes runaway flow, a limit cycle, ageing, or another non-fixed asymptotic regime. No fixed-point equivalence may be claimed.

EXT-U: unresolved sector. This verdict applies when the proof objects do not decide among the preceding branches. Unresolved status is not permission to quotient the sector.

Proposition C.3 (EXT admission contract). *An EXT branch can receive a positive external-equivalence theorem only after all of the following are certified: unique branch type; enlarged state space and RG map; full-sequence asymptotics; complete joint spectrum; protected and dangerous ledgers; reconstructible enlarged observable algebra; external reference frozen before outcome adjudication; and a branch-specific analogue of RG1–RG7.*

Until these conditions hold, the outcome is EXT-UNRESOLVED or a specific failure, never the current PURE/QUOT theorem.

C.6 Strict UMD Universality-Atlas interface

The future Atlas accepts the record

$$\text{Input}(M) = (\text{Thermo}, \text{Carrier}, \mathfrak{F}, \mathfrak{S}, \text{RGOjects}, \text{Observables}, \text{Provenance}). \quad (\text{C.10})$$

It must contain the thermodynamic and locality contracts; carrier rule and persistence status; symmetry, conservation, FDT, absorbing-state, range, and tuning data; spectator poles, gaps, couplings, and protected labels; RG maps, attractors, spectra, and nonlinear fields; observable/response algebra; pre-outcome external-reference freeze; proof-object identifiers; and the first-failure ledger. A missing required field returns UNRESOLVED rather than an inferred value.

The selector is evaluated in the frozen order

$$\begin{aligned} &\text{domain} \rightarrow \text{carrier} \rightarrow \text{fingerprint} \rightarrow \text{spectator} \rightarrow \text{RG asymptotics} \\ &\rightarrow \text{signature} \rightarrow \text{reference} \rightarrow \text{equivalence} \rightarrow \text{verdict}. \end{aligned} \quad (\text{C.11})$$

The first nonpass outcome terminates adjudication. Later evidence cannot retroactively change an earlier branch choice without reopening and refreezing the affected proof object.

The output schema is

$$\begin{aligned} \mathcal{U}_{\text{UMD}}(M) = & (\text{Domain}, \text{Carrier}, \mathfrak{F}, \mathfrak{S}, \Sigma_{\text{RG}}, [\mathbf{T}]_{\text{RG}}, \\ & \text{Reference}, \text{Verdict}, \text{Evidence}). \end{aligned} \quad (\text{C.12})$$

For a positive entry, **Evidence** contains the branch-specific gate certificates and reconstruction contract. Otherwise it contains the first failure code and maximal surviving claim tier.

C.7 Atlas totality, uniqueness, and noncollision

On its declared domain $\mathfrak{D}_{\text{Atlas}}$, the future Atlas must prove:

$$\forall M \in \mathfrak{D}_{\text{Atlas}}, \quad \mathcal{U}_{\text{UMD}}(M) \text{ is defined}; \quad (\text{C.13})$$

$$\mathcal{U}_{\text{UMD}}(M) = U_1 \text{ and } \mathcal{U}_{\text{UMD}}(M) = U_2 \implies U_1 = U_2. \quad (\text{C.14})$$

Noncollision additionally requires that inputs differing in a nonredundant fingerprint, spectral multiplicity, protected datum, dangerous-variable ledger, or reconstruction class cannot receive the same positive entry. Stability requires invariance under admissible analytic coordinate changes, redundant operators, and S1–S7 spectator quotienting.

The present article supplies exactly one complete positive anchor:

$$\boxed{\mathcal{U}_{\text{UMD}}(M) = \mathbf{A}_G(d > 4), \quad M \in \mathfrak{D}_{\text{T5}}^+(d).} \quad (\text{C.15})$$

It may export to the Atlas the fingerprint schema, S1–S7, the EXT registry, normalized signature, equivalence relation, first-failure selector, and (C.15). It does not export as proved the $d = 4$ or $d < 4$ equivalences, Model-B/C branches, nonequilibrium classes, or positive product, coupled, hydrodynamic, or protected EXT entries.

C.8 Formal-boundary and Atlas-interface verdict

The audit proves that $\simeq_{\text{RG}}$ is an equivalence relation on the certified domain; CE1–CE10 separate it from weaker selectors; every EXT sector remains visible under an explicit nonquotient branch contract; and the Atlas input, selector, output, totality, uniqueness, noncollision, and stability requirements are frozen. Therefore

$$\boxed{\text{Appendix C formal-boundary/Atlas-interface audit: PASS.}} \quad (\text{C.16})$$

The appendices are thereby closed without extending the theorem domain or promoting any future Atlas branch to a result of the present article.

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