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THEOREM ON THE EXCLUSION OF HIT TRIPLES AND ABC CONJECTURE

ANNOTATION. The proof of Theorem on the exclusion of hit triples and *abc* conjecture is given by the method of mathematical induction.

KEY WORDS: common divisor, hit numbers, hit triple, mutually prime numbers, our prime numbers, primitive triple, uniform representation.

PRELIMINARY REMARKS. In this article we do not consider 0 as a natural number, but we consider 1 as a our prime number. The sequence of such numbers will be discussed below. We also believe, that if we have agreed to look for solutions to an equation in natural numbers, then the roots of this equation, if they exist, should be uniformly represented as natural numbers.

INTRODUCTION. Let the equation be given

$$\underbrace{A^x}_{a} + \underbrace{B^y}_{b} = \underbrace{C^z}_{c}, \tag{1}$$

where $A, B, C, x, y, z \in \mathbb{N};\ 0 \notin \mathbb{N}$.

The polynomial $F(A,B,C,x,y,z)$ from (1) contradicts the Fundamental theorem of algebra, which requires a polynomial of one variable. However, if we make an transition $F(A,B,C,x,y,z) \Rightarrow F(z) \Rightarrow F'(z)$, then this contradiction is eliminated. The roots of polynomials $F(z)$ and $F'(z)$ are responsible for the general property of the numbers a,b and a,b,c respectively in equation (1).

Any natural number can be represented as n^{ε} (*), where $n, \varepsilon \in \mathbb{N}; 0 \notin \mathbb{N}$. If $\varepsilon = 1$, then such a representation of a natural number should be considered uniform. Among the natural numbers, there are numbers that cannot be represented otherwise than uniformly. We call such numbers our primes. Further by our prime numbers we will understand the numbers, that can be represented as (*) then and only then, when $\varepsilon = 1$. Sequence of such numbers includes in itself: 1, all canonical primes that cannot be decomposed into prime factors, as well as all natural numbers that, although they can be decomposed into prime factors, however can be represented as (*) only uniformly.

AXIOMS OF A UNIFORM REPRESENTATION OF NATURAL NUMBERS. All natural numbers have a uniform representation. All canonical prime numbers can only be represented uniformly. If the representation of a natural number is not uniform, then it is impossible to determine exactly, how this number is represented. A uniform representation is always defined.

Any three numbers A^x, B^y, C^z, that satisfy equation (1), not having a common divisor and ordered in ascending order, where $A, B, C, x, y, z \in \mathbb{N};\ 0 \notin \mathbb{N}$, can be represented as three numbers $a^{\chi}, b^{\delta}, c^{\omega}$, not having a common divisor and ordered in ascending order and satisfying $a^{\chi} + b^{\delta} = c^{\omega}$, where $a, b, c, \chi, \delta, \omega \in \mathbb{N};\ 0 \notin \mathbb{N}$. If $\chi = \delta = \omega = \varepsilon = 1$, then such a representation of a triple of natural numbers should be considered uniform.

We call a uniform representation of three numbers A^x, B^y, C^z, ordered in ascending order, having no common divisor and satisfying (1). In this case the following equalities hold: $A^x = a,\ B^y = b,\ C^z = c$ (2). Unknown a, b, c satisfy $a + b = c$, do not have a common divisor or mutually prime and are an abc triple. If the representation of the three natural numbers a, b, c in the equation $a + b = c$ is non uniform, then it is impossible to determine exactly, how these numbers are represented in $a + b = c$. If the representation of numbers in (1) is uniform, then it is defined.

DEFINITIONS. We call a hit triple of natural numbers the three numbers a, b, c with the above properties, that satisfying equation (1) and inequality $c > \mathrm{rad}(abc)$ (3). The dyad of numbers a, b can also be a hit dyad if the inequality $a + b > \mathrm{rad}(ab(a+b))$ holds (4). With a uniform representation this is not possible because then $abc = \mathrm{rad}(abc)$ (5). In each hit triple there are necessarily numbers whose sequence is formed by the exclusion of our primes from the sequence of natural numbers. We call such numbers hit numbers. The set of hit numbers is formed by the formula (*) by all natural numbers in a row, excluding 1. In each hit triple there is at least one hit number.

We call a primitive triple of natural numbers the three numbers a, b, c that satisfy equation (1) and the inequality $c < \mathrm{rad}(abc)$ (6). The dyad of numbers a, b can also be a primitive dyad if the inequality $a+b<\mathrm{rad}(ab(a+b))$ holds (7).

If in equation (1) the numbers a,b,c do not have a common divisor, then they are either primitive either hit, no third is given (*tertium non datur*). Otherwise these numbers have a common divisor. With a uniform representation hit triples are excluded by the requirement of equality (5).With a uniform representation, the triples of the numbers a,b,c are always primitive, because by virtue of equality (5) inequality (6) always holds. Since natural numbers are formed naturally from counting, they cannot be represented otherwise (8).

THEOREM ON THE EXCLUSION OF HIT TRIPLES. *If the representation of the numbers A^x, B^y, C^z in (1) is undefined and they satisfy (2) and the inequality $c<\mathrm{rad}(abc)^2$, then the triple of numbers a,b,c cannot be a hit.*

PROOF. Suppose, that the hit triples exist, satisfing (3). If equation (1) has solutions in natural numbers, then the following strict equality always holds:

$$\frac{C}{\sqrt[z]{A^x+B^y}} \equiv 1. \tag{9}$$

If $A^x+B^y=\alpha,\ \alpha\in\mathbb{N}$, then and $C^z=\alpha$. If α is the real part of the complex number $\gamma=\alpha+i\beta$, then and $C^z=\gamma$, provided that $\gamma\neq 0,\ \beta=0$. If $z\in\mathbb{N},\ \gamma\in\mathbb{C}$, then the root of the degree $z$of γ is the solution of the equation $C^z=\gamma$. Using the Moivre formula, you can write:

$$\begin{aligned} C=\sqrt[z]{\gamma}=\sqrt[z]{|\gamma|}(\cos\frac{\varphi+2\lambda\pi}{z}+i\sin\frac{\varphi+2\lambda\pi}{z})= \\ =\sqrt[z]{\sqrt[2]{\alpha^2+\beta^2}}(\cos\frac{\varphi+2\lambda\pi}{z}+i\sin\frac{\varphi+2\lambda\pi}{z}). \end{aligned} \tag{10}$$

Here $|\gamma|$ is the module γ, $\varphi=\arg\gamma$ is the main value of $\mathrm{Arg}\gamma$, $\lambda=0,1,2...z-1$ $(0\leq\lambda\leq z-1)$ or $\lambda=z-1$ (11).

In the case when $\alpha=A^x+B^y;\ \beta=0;\ A,B,C,x,y,z\in\mathbb{N}$ (10) takes the following form:

$$C = \underbrace{\sqrt[z]{A^x + B^y}}_{\text{Mod}C} \times \underbrace{(\cos\frac{\varphi + 2\lambda\pi}{z} + i\sin\frac{\varphi + 2\lambda\pi}{z})}_{\text{Arg}C} \Rightarrow$$

$$\Rightarrow C = \text{Mod}C \times \text{Arg}C \Rightarrow \text{Mod}C \times F(z) \Rightarrow \text{Mod}C \times F'(z) \Rightarrow$$

$$\Rightarrow C < \sqrt[z]{A^x + B^y} \times F'(z). \qquad (12)$$

Here $F(z)$ is a certain polynomial, the two roots of which are responsible for the properties of a pair of numbers a, b in the equation $a + b = c$. The polynomial $F'(z)$ derived from it gives three roots that are responsible for the properties of the triple of numbers a, b, c in the same equation.

Equation (12) can be equivalent to (1) only in the case of $\text{Arg}C = 1$. Therefore first we define $F(z)$. Hence follows $F'(z)$. Divide (12) by $\text{Mod}C = \sqrt[z]{A^x + B^y}$, we will take into account (9), we get:

$$1 = \underbrace{(\cos\frac{\varphi + 2\lambda\pi}{z} + i\sin\frac{\varphi + 2\lambda\pi}{z})}_{\text{Arg}C} \Rightarrow \text{Arg}C \Rightarrow F(z) \Rightarrow F'(z). \qquad (13)$$

The left part (13) is equivalent to the following system:

$$\begin{cases} \cos\dfrac{\varphi + 2\lambda\pi}{z} = 1, \\ \sin\dfrac{\varphi + 2\lambda\pi}{z} = 0. \end{cases} \qquad (14)$$

From (14) we have $(\varphi + 2\lambda\pi)/z = 2\lambda\pi$ (15). Multiply both sides (15) by z, find: $\varphi = 2\pi(\lambda z - \lambda)$ (16). Divide both parts of equation (16) by 2π, we get: $\varphi/(2\pi) = \lambda z - \lambda$ (17). Since $z \in \mathbb{N}$, φ must be a multiple of 2π or equal to 0. Otherwise there are no solutions to equation (17) on $\mathbb{N}$. Note that $\varphi = 0$ follows from $\cos\varphi = \alpha/\sqrt{\alpha^2 + \beta^2}$ for $\beta = 0$, since in this case $\cos\varphi = \alpha/\sqrt{\alpha^2} = \alpha/\alpha = 1$ and $\varphi = 0$. Therefore, substituting $\varphi = 2\lambda\pi$ into the left part (17), we get: $n\lambda - 2\lambda = 0$ (18).

After substituting (11) into (18), we have: $F(z) = z^2 - 3z + 2 = 0$ (19). This polynomial can be written otherwise:

$$F(z)=z^2-3z+2=\underbrace{(z-1)}_{A^x}\times\underbrace{(z-2)}_{B^y}\Rightarrow A^x\times B^y=ab\geq \mathrm{rad}(ab). \qquad (20)$$

The numbers a,b in (20) here are $(1,2)$. They are adjacent. It is known that adjacenting natural numbers are always mutually prime. Any two natural numbers that are unequal to each other and do not have a common divisor, when added together give a triple of mutually prime numbers (21).

From (20) and (21) it follows:

$$F(z)\Rightarrow F'(z)=\underbrace{(z-1)}_{A^x}\times\underbrace{(z-2)}_{B^y}\times\underbrace{(z-3)}_{C^z}\Rightarrow A^x\times B^y\times C^z=abc\geq \mathrm{rad}(abc). \quad (22)$$

There is only one triple of adjacent natural numbers $a,b,c,$ satisfying the equation $a+b=c.$ This is $(1,2,3)$. However by virtue of (21) this attribute is weaker for numbers $a,b,c,$ than the fact, that they are mutually prime numbers.

The dyad $(a,b)=(1,2)$ and the triple $(a,b,c)=(1,2,3)$ consist of numbers, that are the first numbers of the sequence of our primes (*sic erat scriptum*).

Let us now return to (12). If we raise (12) to the power of $z,$ we get:

$$C^z<(A^x+B^y)\times F'(z)^z. \qquad (23)$$

Let's replace the variables according to (2). Suppose that the representation of numbers in (23) is uniform, that is, the condition $z\Rightarrow\varepsilon=1$ is satisfied. Such a transition is possible only to a uniform representation, because, as we noted above, there are no hit triples with a uniform representation. Taking into account (22) we get an obvious inequality: $c<(a+b)\times abc$ (24).

The requirement of inequality (24) is suitable for the uniform representation of numbers in (1). If the representation is not defined, then such a requirement becomes redundant. To overcome this redundancy, we need to introduce a constant K_ε, taking into account which the inequality (24) takes the following form: $c\leq K_\varepsilon(a+b)\times abc$ (25), where $K_\varepsilon\geq 1/(abc)$ (26). It follows from (26) that always $K_\varepsilon<1$. The value of $K_\varepsilon>1$ does not make any sense.

If the representation of numbers in (1) is undefined, then taking into account

(7) and (22) the inequality (25) can be transformed as follows:

$$c < K_{\varepsilon}\mathrm{rad}(ab(a+b))\mathrm{rad}(abc) = K_{\varepsilon}\mathrm{rad}(abc)\mathrm{rad}(abc) = K_{\varepsilon}(\mathrm{rad}(abc))^2. \quad (27)$$

Suppose that $K_{\varepsilon} = 1$. In this case (27) takes the following form:

$$c < (\mathrm{rad}(abc))^2. \quad (28)$$

Since (28) contradicts (3), the theorem is proved ⊗.

CONSEQUENCE. For any $\varepsilon > 0$ there is a constant $0 < K_{\varepsilon} \leq 1$, at which the inequality $c \leq K_{\varepsilon}(\mathrm{rad}(abc))^{1+\varepsilon}$ holds for any mutually prime numbers a,b,c.

We apply the method of mathematical induction. It is established that (28) is true. This will be the base of induction. Let us represent (28) in the form:

$$c < \mathbf{1} \times (\mathrm{rad}(abc))^{\mathbf{2}}. \quad (29)$$

Applying the induction transition $\mathbf{1} \Rightarrow K_{\varepsilon}; \mathbf{2} \Rightarrow 1+\varepsilon$, we obtain:

$$c \leq K_{\varepsilon}(\mathrm{rad}(abc))^{1+\varepsilon}. \quad (30)$$

CONCLUSION. The exclusion of hit triples from (1) allows to determine all hit numbers by the formula (*) and consequently all hit triples (*quid est exclusus est definitum*).

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