On the Asymptotics of the Schrödinger Equation Solutions and the Euler Model of an Ideal Fluid

V. V. Khablov

Novosibirsk State Technical University, 20 K. Marx prospect, Novosibirsk 630073, Russia

E-mail: `xablov@corp.nstu.ru; hablovvladislav@gmail.com`

Abstract

In this paper we analyze the asymptotics of the Schrödinger equation solutions with respect to a small parameter $\hbar$. It is well known, that short-wave asymptotics to solutions of this equation leads to the pair of equations— the Hamilton–Jacobi equation for the phase and the continuity equation. These equations coincide with the ones for the potential flows of an ideal fluid. The wave function is invariant with respect to the complex plane rotations group, and the asymptotics is constructed as a point-dependent action of this group on some function that is found by solving the transfer equation. It is shown in the paper, that if the Heisenberg group is used instead of the rotation group, then the limit of the equations solutions with $\hbar$ tending to zero leads to the equations for vortex flows of an ideal fluid in a potential field of forces. If the original Schrödinger equation is nonlinear, then equations for barotropic processes in an ideal fluid are obtained.

Keywords: Schrödinger equation, Euler equations, short-wave asymptotics, quasi-classical approximation, quasi-classical limit

The structure of the message

Quasi-potential To clarify the result of the work, first in **1** section it is noted, that the class of the external differential of the energy-momentum form $\alpha = pdx - Hdt$, used in classical Hamiltonian mechanics, does not exceed n on an arbitrary integral surface of dimension $n+1$. It follows from this, that on an arbitrary integral surface of dimension four in the expanded phase space R^7, the form $d\alpha$ is decomposable. For the form α, this means, that it can be represented on this surface as $dS - mdn$ with some functions S, m and n. This triplet of scalar functions is referred to as the energy-momentum form α in the work. There is given an example in which $\alpha = vdx - \left(\frac{v^2}{2} + U(x,t)\right)dt$. In this case the quasi-potential satisfies a system of equations, which is equivalent to the Euler system of equations of an ideal fluid, and the triple S, m, and n are known as Klebsch potentials.

Heisenberg group In the next **2** section the main object is entered—the Schrödinger equation, the solution of which is the function ψ of the point (x, y, z, t) with values in the Hilbert space $L^2(R)$, and the Heisenberg group H. We write out solutions of the Schrödinger equation by using elements of the group H.

Asimptotics In the last **3** section, the behavior of solutions of the Schrödinger equation with respect to h tending to zero is investigated, and, as a consequence, it is shown there, that the asymptotics obey the equations for the quasi-potential of the Euler model of an ideal fluid.

1 Quasi-potential

Let's take the index form for partial derivatives notation for brevity: f_t instead of $\frac{\partial f}{\partial t}$, f_{yy} instead of $\frac{\partial^2 f}{\partial y^2}$, etc., Δ— Laplace operator on variables (x, y, z).

Further the presentation has a local character, that is, it is assumed, that the variables belong to some sufficiently small neighborhood of a fixed point, in which all the functions under consideration and their derivatives are continuous up to the order used. The goal here is to obtain a more convenient form of the Euler model of an ideal fluid. It is made by introducing the "quasi-potential". The nature of the objects is general, and it is necessary to explicitly reproduce partially well-known facts, "folklore", putting them in order.

1.1 On the energy-momentum form class

Consider a Hamilton system with the Hamiltonian
$H(x, p, t)$, defined in the expanded phase space R^{2n+1}

$$\dot{x}_i = H_{p_i}; \quad \dot{p}_i = -H_{x_i}; \quad i = 1, 2. \dots, n.$$

Let Λ^{n+1} be a smooth integral surface of dimension $n + 1$, with respect to which the Hamiltonian vector field $\boldsymbol{\partial} = \frac{\partial}{\partial t} + H_p \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial p}$ (with implied summation) is tangent. Denote α^* the narrowing of the form α by Λ^{n+1}. Next, the dot above the function name is the result of the field $\boldsymbol{\partial}$ applying. Takes place

Observation. *If Λ^{n+1} is a smooth integral surface for $\boldsymbol{\partial}$, then the class of the differential form* [1] *$d\alpha^*$ does not exceed n. In particular, if n=3, then the form $d\alpha^*$ is decomposable, and hence there is a triple S, m, n of such smooth functions, defined on Λ^4 surface, that $ds - mdn = pdx - Hdt$. These functions satisfy the system of equations*

$$\dot{m} = \dot{n} = 0; \ \dot{S} = pH_p - H.$$

Indeed, the form $d\alpha = dp \wedge dx - dH \wedge dt$ is, according to E. Kartan, absolute integral invariant of the field $\boldsymbol{\partial}$ ([1],[2]), that is, on Λ^{n+1}surface the inner product $i_{\boldsymbol{\partial}} d\alpha = 0$. The field $\boldsymbol{\partial}$ is not degenerated by Λ^{n+1}. It follows from this, that the class of the form $(d\alpha)^*$, narrowed to the surface Λ^{n+1}, does not exceed n. In the case $n = 3$ form $(d\alpha)^*$ is decomposable. This means that there are a pair of such functions m and n defined on Λ^4 surface, that $(d\alpha)^* = d\alpha^* = dn \wedge dm$, and $\alpha^* = dS - mdn$ for some function S. Assuming that $(d\alpha)^* \neq 0$ on Λ^4 surface, then from the equality $0 = i_{\boldsymbol{\partial}} d\alpha = i_{\boldsymbol{\partial}}(d\alpha)^* = i_{\boldsymbol{\partial}}(dn \wedge dm)$ it now follows that $\dot{m} = \dot{n} = 0$, and from the fact that $\alpha = dS - mdn$ follows the equality $\dot{S} = pH_p - H$.

1.2 Integral surface building and quasi-potential

On the reduction of the Cauchy problem for a quasi-potential to the Cauchy problem for a system of ordinary differential equations.

Let the Hamiltonian $H(x, t, p)$ be given. In accordance with the above, the following possible way of solving a system of equations for a quasi-potential is formed:

1) The initial manifold Λ_0^3 of 3 dimension in R^7 is constructed according to the momentum distribution $p(x, 0)$ given at $t = 0$.

2) Using the flow specified by the vector field $\boldsymbol{\partial}$, a manifold $\Lambda^4 = \bigcup\limits_{t \geq 0} \Lambda_t^3$ is constructed from the initial manifold Λ_0^3.

3) For $t = 0$ we define S, m, n from the equality $dS - mdn = pdx$. Field $\boldsymbol{\partial}$ is tangent to the surface Λ^4, and, we find the quasi-potential $S,\ m,\ n$, defined on Λ^4 surface, by solving the equations $\dot{m} = \dot{n} = 0$, $\dot{S} = pH_p - H$.

Thus, this approach is a double of the characteristics method for solving the Hamilton–Jacobi equation, and the variety Λ^4 plays the role of a Lagrangian manifolds Λ in constructing the velocity potential of a vortex-free flow [3].

We also note the property of the gauge invariance of the resulting system for the quasi-potential. It consists in the fact, that the existing distribution of pulses on the variety Λ^4, the quasi-potential $S,\ m,\ n$ is not uniquely defined: any transformation of the triplet S, m, n, leaving the form $dS - mdn = dS' - m'dn'$ unchanged, leads us to the same system of equations for the new quasi-potential S', m', n'.

Consider an example. Moving on to the notation usually accepted ($p \leftrightarrow v$), take the Hamiltonian $H = U(x, y, z, t) + \frac{v^2}{2}$, $\alpha = vdx - \left(\frac{v^2}{2} + U(x, t)\right)dt$. Euler equations for fluid flows in the potential field of forces ([4], [5]) are

$$\begin{cases} \dot{v} + \nabla U(x, y, z, t) = 0; \\ \rho_t + \operatorname{div}(\rho v) = 0. \end{cases}$$

($U(x, y, z, t) = U(\rho)$ for barotropic processes.)
The first three equations of this system are Hamiltonian, and the **Observation** given above The first three equations of this system are Hamiltonian, and the Observation given above refers to them. Suppose that the surface Λ^4 is diffeomorphically projected into $R^4_{(x,t)}$. In this case, we come to the Clebsch representation, known in hydrodynamics ([6], 167; [7], Sect. 29) solutions of the Euler equations.

We formulate explicitly the sentence used further ([6], 167; [7], Sect. 29).

Proposition. *The vector function v and the scalar ρ are Euler equations solution* *if and only if there exist such functions* $S,\ m,\ n$, *for which* $\nabla S - m\nabla n \equiv v$ *and*

$$\begin{cases} S_t - mn_t + \dfrac{v^2}{2} + U(x,y,z,t) \equiv 0; \\ m_t + (v\cdot\nabla)m = 0; \\ n_t + (v\cdot\nabla)n = 0; \\ \rho_t + \operatorname{div}(\rho v) = 0. \end{cases}$$

($U(x,y,z,t) = \mathcal{P}(\rho)$ *for barotropic processes*.)

The triple of functions $S,\ m,\ n$ will be called *a quasi-potential for the original Euler equations solution*, and the obtained system—a system of equations for a quasi-potential. Such a quasi-potential by **Proposition** always exists for Euler equations solution, its existence has a local character.

2 Schrödinger equation and Heisenberg group

Consider the Schrödinger equation

$$i\hbar\psi_t + \frac{\hbar^2}{2M}\Delta\psi - U(x,y,z,t)\psi = 0.$$

It is usually assumed, that ψ is a complex-valued state function, and the square of its module $|\psi|^2$ is interpreted as the density of the distribution probability for a particle of mass M to be at time t in the point with coordinates (x,y,z).

If we proceed from the requirement, that the state function reflects the presence of spin, or some charge and the like, then within the framework of the idea of introducing calibration fields, we can consider the function ψ as a vector of the unitary representation space of the corresponding Lie group G and, accordingly, its Lie algebra $\mathfrak{a}$. Further steps are associated with the construction of gauge invariant equations and so on. Another option is—building a model, "using" group asymptotically. Thus, for the classical short-wave asymptotics of solutions of the Schrödinger equation, the group $G = U(1) = \{e^{iS}\}$ and the one-dimensional space C^1 of its representation are used. The construction of asymptotics means, in particular, the determination of the dependence of the group element e^{iS} on the point (x,y,z,t). If the dimension of the unitary space of the ψ representation is greater than one, but finite, then the form of the equation Schrödinger and the interpretation of it's solutions practically do not change. What is convenient—with this approach, we can use all the results on the solvability of the Cauchy problem for the Schrödinger equation in suitable functional spaces. There are no problems with solutions interpreting in both the classical and generalized sense.

Construction the quasi-classical approximation implies the use of the classical Hamiltonian $H = U(x,y,z,t) + \frac{v^2}{2}$ and the integral surface Λ^4 of the corresponding Hamiltonian system. The surface Λ^4 is Lagrangian, that is, the form $d\alpha$ vanishes on it. The requirement for surface Λ^4 to be Lagrangian is related to the group $G = U(1)$, used for the construction.

It is proposed to abandon the requirement of a surface Λ^4 to be Lagrangian and reduce these requirements for Λ^4 to the minimum for the classical Hamiltonian mechanics, that is, to the decomposability of the form $d\alpha$ on Λ^4, what is necessary, according to the **Observation** of the previous section. To represent solutions of the Schrödinger equation and construct the asymptotics with $\hbar$ tending to zero, a wider Lie group H is used instead of the group $G = U(1)$. It is the Heisenberg group of upper triangular 3×3 matrices, that does not have finite-dimensional unitary representations. Unitary representations of the group H are possible in an infinite-dimensional Hilbert space, so further ψ—is a function of variables (x, y, z, t) with values in space $L^2(R)$, $|\psi|$—the norm in this space defined by the scalar product

$$\langle \psi_1, \psi_2 \rangle = \int_R \psi_1(\xi)\overline{\psi}_2(\xi)d\xi.$$

We will assume, that the Schrödinger equation solution ψ is defined, has classical partial derivatives included in the equation, and they are continuous. Next, in the Schrödinger equation, $M = 1$.

So, the symbol H means the Heisenberg group of triangular matrices $g(m, n, S) = \begin{pmatrix} 1 & m & S \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$. Its unitary representation is realized by the left action of an arbitrary element $g(S, m, n) \in H$ on the elements of $u \in L^2(R)$ by the formula [8]

$$(g(S, m, n)u)(\xi) = e^{(S+n\xi)i}u(\xi + m).$$

In these notations, $(g(S, m, n))^{-1} = g(-S + mn, -m, -n)$. For the selected representation, the basis of the Lie algebra of the group H can be chosen as follows: $\left\{i,\ p = \frac{d}{d\xi},\ q = i\xi\right\}$.

Let (S, m, n) are the scalar functions of variables (x, y, z, t), and u is the function of variables (x, y, z, t) with a values in $L^2(R)$. In these notations it is convenient when calculating derivatives to leave elements of $g(S, m, n) \in H$ on the left. So, for example, the result of differentiation by the variable t looks like:

$$(g(S, m, n)u)_t = g(S, m, n)\left(m_t p + n_t q + i(S_t - mn_t)\right)u + g(S, m, n)u_t. \quad (1)$$

It also turns out (with the notation $v = \nabla S - m\nabla n$)

$$\nabla\left(g(S, m, n)u\right) = g(S, m, n)\left(iv + (\nabla m)\, p + (\nabla n)\, q + \nabla\right)u.$$

Now we calculate $\Delta\left(g(S, m, n)u\right)$ by the same way

$$\begin{aligned} \Delta(g(S, m, n)u) &= \operatorname{div}(g(S, m, n)(iv + (\nabla m)p + (\nabla n)q + \nabla)u) = \\ &= g(S, m, n)(iv \cdot ((iv + (\nabla m)p + (\nabla n)q + \nabla))u + \\ &+ g(S, m, n)(((\nabla m)p + (\nabla n)q) \cdot (iv + (\nabla m)p + \\ &+ (\nabla n)q))u + g(S, m, n)(((\nabla m)p + (\nabla n)q) \cdot \nabla)u + \\ &+ g(S, m, n)\operatorname{div}((iv + (\nabla m)p + (\nabla n)q + \nabla)u) \end{aligned}$$

If we denote $m_v = (v \cdot \nabla) m$, $n_v = (v \cdot \nabla) n$, then

$$\begin{aligned}\Delta (g(S,m,n)u) = g(S,m,n)\left(-v^2 + im_v p + in_v q + i(v\cdot\nabla)\right)u+\\ + g(S,m,n)\left(im_v p + in_v q + ((\nabla m)\,p + (\nabla n)\,q)^2\right)u+\\ + g(S,m,n)\,((\nabla m\cdot\nabla)\,p + (\nabla n\cdot\nabla)\,q)\,u+\\ + g(S,m,n)\,(i\,\mathrm{div}\,v + (\Delta m)\,p + (\Delta n)\,q + \Delta)\,u+\\ + g(S,m,n)\,((iv + (\nabla m)\,p + (\nabla n)\,q)\cdot\nabla)\,u\end{aligned}$$

Combining like terms, we obtain an expression for the Laplacian:

$$\begin{aligned}\Delta (g(S,m,n)u) = g(S,m,n)\Big(((({\nabla m})\,p + (\nabla n)\,q)^2 - v^2+\\ + 2im_v p + 2in_v q + (\Delta m)\,p + (\Delta n)\,q + i\,\mathrm{div}\,v)u+\\ + ((2\,(\nabla m)\,p + 2\,(\nabla n)\,q + 2iv)\cdot\nabla)\,u + \Delta u\Big)\end{aligned} \quad (2)$$

3 Generalized quasi-classical asymptotics

Let us proceed to the construction of the formal asymptotics of solutions of the Schrödinger equation. Let's introduce the notation $\varepsilon = \sqrt{\hbar}$, $\hbar$—"the Planck constant", a small parameter, that we will aim at zero, while monitoring the behavior of the solution $\psi[\varepsilon]$.

We are looking for the solution $\forall\varepsilon > 0$ in the form

$$\psi[\varepsilon](x,y,z,t) = g\left(\varepsilon^{-2}S(x,y,z,t), \varepsilon^{-1}m(x,y,z,t), \varepsilon^{-1}n(x,y,z,t)\right)u(x,y,z,t),$$

where $u(x,y,z,t)$ is a function with values in $L^2(R)$, $g \in H$.

Further it assume that u belongs to L—the subspace of $L^2(R)$, which is the common part of the domains of the operators p^2 and q^2 with the graph norms. Also we will assume, that the Schrödinger equation solution ψ is defined, has classical partial derivatives included in the equation, and they are continuous as functions with values in L.

Proposition 1 *The function $\psi[\varepsilon](x,y,z,t)$ is an asymptotic solution of the Schrödinger equation up to the second order with respect to ε tending to zero if and only if the functions S, m, n and u are the solution of a system of equations*

$$\begin{cases} S_t - mn_t + \frac{v^2}{2} + U(x,y,z,t) = 0;\\ m_t + m_v = 0;\\ n_t + n_v = 0;\\ iu_t + \frac{i}{2}(\mathrm{div}\,v)u + i\,(v\cdot\nabla)\,u + \frac{1}{2}\,((\nabla m)\,p + (\nabla n)\,q)^2\,u = 0.\end{cases} \quad (3)$$

Proof. Substitute the $\psi[\varepsilon]$ ansatz into the Schrödinger equation and apply to both parts of the equation $\left(g\left(\varepsilon^{-2}S, \varepsilon^{-1}m, \varepsilon^{-1}n\right)\right)^{-1}$. Using (1) and (2), we arrive at the following equality:

$$\begin{aligned}
&\Big(-(S_t - mn_t) - \frac{v^2}{2} - U(x,y,z,t)\Big)u+\\
&\quad + \varepsilon i((m_t + m_v)\,p + (n_t + n_v)\,q)u+\\
&\quad + \varepsilon^2 i\Big(\Big(\frac{\partial}{\partial t} + \frac{1}{2}\operatorname{div} v\Big)u + (v\cdot\nabla)\,u - i\,((\nabla m)\,p + (\nabla n)\,q)^2\,u\Big)+ \qquad (4)\\
&\quad + \frac{\varepsilon^3}{2}((\Delta m)\,p + (\Delta n)\,q + 2\,(((\nabla m)\,p + (\nabla n)\,q)\cdot\nabla))u+\\
&\quad + \varepsilon^4 \Delta u = 0,
\end{aligned}$$

where $v = \nabla S - m\nabla n$. The conditions ensuring the fulfillment of this equality up to the second order of smallness with respect to ε tending to zero coincide with the system equations of the formulated sentence. ■

We obtain a consequence of the last equation of this system. Denote $\rho = |\psi|^2 = |u|^2$. Considering, that the first three equalities take place (asymptotics up to the first order by ε), multiply $\forall(x,y,z,t)$ both parts of the fourth equation for asymptotics scalar by u (that is, $\langle\ldots,u\rangle$) and extract the imaginary part as a result. The first three terms give the expression $\rho_t + \operatorname{div}(\rho v)$. For the last one $\frac{1}{2}\,((\nabla m)\,p + (\nabla n)\,q)^2\,u$ we get

$$\begin{aligned}
&\operatorname{Im}\langle((\nabla m)\,p + (\nabla n) q)^2 u, u\rangle =\\
&= -\frac{i}{2}\int_R ((\nabla m)\,p + (\nabla n)\,q)^2\,u(\xi)\overline{u}(\xi)d\xi+\\
&+ \frac{i}{2}\int_R \overline{((\nabla m)\,p + (\nabla n)\,q)^2\,u(\xi)}u(\xi)d\xi =\\
&= \big|\text{integration by parts}\big| =\\
&= \frac{i}{2}\int_R ((\nabla m)\,p + (\nabla n)\,q)\,u(\xi)\overline{((\nabla m)\,p + (\nabla n)\,q)\,u(\xi)}d\xi-\\
&- \frac{i}{2}\int_R \overline{((\nabla m)\,p + (\nabla n)\,q)\,u(\xi)}\,((\nabla m)\,p + (\nabla n)\,q)\,u(\xi)d\xi = 0.
\end{aligned}$$

We have the continuity equation for $\rho : \rho_t + \operatorname{div}(\rho v) = 0$.

Together with the first three equations satisfied by the asymptotics, this gives a system of Euler equations for the quasi-potential

$$\begin{cases}
S_t - mn_t + \frac{v^2}{2} + U(x,y,z,t) = 0;\\
m_t + m_v = 0;\\
n_t + n_v = 0;\\
\rho_t + \operatorname{div}(\rho v) = 0.
\end{cases}$$

From here we conclude:

Theorem 1 *If the function*

$$\psi[\varepsilon](x,y,z,t) = g\left(\varepsilon^{-2}S, \varepsilon^{-1}m, \varepsilon^{-1}n\right)u(x,y,z,t)$$

is an asymptotic solution of the Schrödinger equation up to the second order with respect to ε tending to zero, then the functions $S,\ m,\ n$ and $\rho = |u|^2$ satisfy the system of Euler equations for a quasi-potential, describing the flow of an ideal fluid in a potential field of forces. In particular, if the potential $U = \mathcal{P}\left(|\psi|^2\right)$, then these equations describe vortex flows for barotropic processes.

$$\begin{cases} S_t - mn_t + \dfrac{v^2}{2} + \mathcal{P}(\rho) = 0; \\ m_t + m_v = 0; \\ n_t + n_v = 0; \\ \rho_t + \operatorname{div}(\rho v) = 0 \end{cases} \tag{5}$$

■

It is not difficult to write out conditions for $\psi[\varepsilon]$ that ensure the fulfillment of the Schrödinger equation with an accuracy higher, than the second power of the parameter ε. To do this, apply $\left(g\left(\varepsilon^{-2}S, \varepsilon^{-1}m, \varepsilon^{-1}n\right)\right)^{-1}$ to both parts of the Schrödinger equation, in which was substituted $\psi = \psi[\varepsilon]$. Grouping coefficients at powers of ε as in (4), and using short notation $\mathcal{L}_2$ and $\mathcal{L}_3$, we get

$$\begin{aligned} &\left(-(S_t - mn_t) - \frac{v^2}{2} - U(x,y,z,t)\right)u + \varepsilon i((m_t + m_v)\,p+ \\ &\qquad + (n_t + n_v)\,q)u + \varepsilon^2 \mathcal{L}_2 u + \varepsilon^3 \mathcal{L}_3 u + \varepsilon^4 \Delta u = 0. \end{aligned} \tag{6}$$

Let's represent in (6) $u(x,y,x,t)$ as $u = u_0 + \varepsilon u_1 + \ldots + \varepsilon^n u_n$

Proposition 2 *For $n = 1$, the triple $S,\ m,\ n$ is an asymptotic solution of the Schrödinger equation up to the third order with respect to ε tending to zero if and only if the functions $S,\ m,\ n$ and u are the solution of a system of equations*

$$\begin{cases} S_t - mn_t + \dfrac{v^2}{2} + U(x,y,z,t) = 0; \\ m_t + m_v = 0; \\ n_t + n_v = 0; \\ \mathcal{L}_2 u_0 = 0; \\ \mathcal{L}_2 u_1 + \mathcal{L}_3 u_0 = 0. \end{cases}$$

For $n > 1$, the triple $S,\ m,\ n$ is an asymptotic solution of the equation Schrödinger up to $n+2$ order with respect to ε, tending to zero if and only if the functions $S,\ m,\ n$ and u are the solution of a system of equations

$$\begin{cases} S_t - mn_t + \dfrac{v^2}{2} + U(x,y,z,t) = 0; \\ m_t + m_v = 0; \\ n_t + n_v = 0; \\ \mathcal{L}_2 u_0 = 0; \\ \mathcal{L}_2 u_1 + \mathcal{L}_3 u_0 = 0; \\ \mathcal{L}_2 u_2 + \mathcal{L}_3 u_1 + \Delta u_0 = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathcal{L}_2 u_n + \mathcal{L}_3 u_{n-1} + \Delta u_{n-2} = 0. \end{cases} \tag{7}$$

The proof consists in equating to zero the coefficients for all degrees ε of the left side of the equation (6) ■

Corollary 1 *If for $n \geq 0$ the function $\psi[\varepsilon](x, y, z, t)$ with $u = u_0 + \varepsilon u_1 + \ldots + \varepsilon^n u_n$ is an asymptotic solution of the Schrödinger equation up to $n + 2$ order with respect to ε tending to zero, then*

$$\left(-(S_t - mn_t) - \frac{v^2}{2} - U(x, y, z, t)\right) u + \varepsilon i((m_t + m_v)\, p + (n_t + n_v)\, q)u+ \\ + \varepsilon^2 \mathcal{L}_2 u + \varepsilon^3 \mathcal{L}_3 u + \varepsilon^4 \Delta u = O(\varepsilon^{n+3}). \tag{8}$$

Proof. If we multiply the first equation of the system (7) by -1, the second — by εip, the third — by εiq, the fourth — by ε^2, the fifth — by ε^3, and so on, then add everything up, then you get the ratio (8). ■

Corollary 2 *If the function*

$$\psi[\varepsilon](x, y, z, t) = g\left(\varepsilon^{-2} S, \varepsilon^{-1} m, \varepsilon^{-1} n\right) u(x, y, z, t)$$

is an asymptotic solution of the Schrödinger equation up to $n + 2$ of order ($n > 1$) with respect to ε tending to zero, then the functions S, m, n and $\rho = |u|^2$ satisfy a system of Euler equations for a quasi-potential describing the flow of an ideal fluid in a potential field of forces.

$$\begin{cases} S_t - mn_t + \dfrac{v^2}{2} + U(x, y, z, t) = 0; \\ m_t + m_v = 0; \\ n_t + n_v = 0; \\ \rho_t + \operatorname{div}(\rho v) = O(\varepsilon^{n+1}). \end{cases}$$

In particular, if the potential $U = \mathcal{P}\left(|\psi|^2\right)$, then these equations describe vortex flows for barotropic processes.

$$\begin{cases} S_t - mn_t + \dfrac{v^2}{2} + \mathcal{P}(\rho) = 0; \\ m_t + m_v = 0; \\ n_t + n_v = 0; \\ \rho_t + \operatorname{div}(\rho v) = O(\varepsilon^{n+1}). \end{cases}$$

The proof steps repeat the output of **Theorems 1**.

Conclusions

1. For the classical object of Hamiltonian mechanics — differential form energy-momentum $\alpha = pdx - Hdt$, the emphasis is placed on the fact that the class of external differential of this form does not exceed n on an arbitrary integral surface of dimension $n + 1$. It follows from this that for the case $n = 3$ on an arbitrary integral surface of dimension four in an expanded phase space R^7 the form $d\alpha$ is decomposable. For the form α, this means the possibility of its representation on this surface as $\alpha = dS - mdn$ with some functions S, m and n. This triple of scalar functions is called the quasipotential of the energy–momentum form α. For the case of the form $\alpha = vdx - \left(\frac{v^2}{2} + U(x,t)\right)dt$ the quasi-potential of S, m and n satisfies system of equations equivalent to the system of Euler equations of an ideal fluid and gives Clebsch representation of solutions to the Euler system (**Proposition** in Section **1**).

2. As the initial quantum objects, it is proposed to use the Schrödinger equation for the wave function ψ with values in the $L^2(R)$ Hilbert space. This makes it possible to use a non-trivial unitary representation of the Heisenberg group in this space and use it to construct asymptotic solutions of the Schrödinger equation. The proposed construction generalizes the widely used quasi-classical approximation. The generalization form $\psi[\varepsilon](x, y, z, t)$ is given at the beginning section **3**.

3. It was proved (**Proposition 1**) that the first three conditions for the existence of an asymptotic solutions of the Schrödinger equation in the form $\psi[\varepsilon](x, y, z, t)$ coincide with the corresponding equations of the system for the quasi-potential, and the consequence of the last condition is continuity equation. **Theorem 1** generalizes this statement to the case of dependence $U = \mathcal{P}(|\psi|^2)$ potential in the Schrödinger equation. Asymptotic triple S, m and n in this case leads to equations for barotropic processes. In Proposition 2 and Corollaries, ways of increasing the accuracy of the asymptotics to powers $\varepsilon = \sqrt{\hbar}$ above the second.

4. Note that from the above results it follows the possibility of constructing an "incomplete" asymptotics of solutions of the Schrödinger equation. Meaning that you can find S, m, n and $\rho = |\psi|^2$, that is, the action of the elements of the Heisenberg group on the initial wave function and module of that function without finding the function itself. This can be done by first finding solution of the Euler system of equations

$$\begin{cases} \dot{v} + \nabla U(x, y, z, t) = 0; \\ \rho_t + \operatorname{div}(\rho v) = 0. \end{cases}$$

($U(x, y, z, t) = \mathcal{P}(\rho)$ for barotropic processes.) Then find S, m and n from the condition $d\alpha = dS - mdn$. Alternatively search for S, m, n and ρ from the equations for the quasi-potential.

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