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abc CONJECTURE, z LEMMA AND HIT CONJECTURES

ANNOTATION. Proofs of z lemma and Pillai's conjecture are given.

KEY WORDS: abc triple, coprime numbers, canonical prime factors, common prime divisor, finite closed interval, hit conjectures, hit numbers, hit triples, ordered in ascending order numbers, perfect power.

PRELIMINARY REMARKS. To properly understand this preprint, you need to familiarize yourself with another of our preprints [Kh1]. With the sign $\otimes$ we indicate that the statement is proved. The equations in hit conjectures are interpreted by us as equations of the form $\underbrace{A^x}_{a}+\underbrace{B^y}_{b}=\underbrace{C^z}_{c}$ (*), where $A,B,C,x,y,z\in\mathbb{N}$.

INTRODUCTION. For any $\varepsilon>0$ (1) there exists a constant K_ε, such that $0<K_\varepsilon\le 1$ (2), at which for any three coprime and ordered in ascending order numbers $a,b,c\in\mathbb{N}$, satisfying equation $a+b=c$ (3), the inequality holds:

$$c\le K_\varepsilon(\mathrm{rad}(abc))^{1+\varepsilon}. \tag{4}$$

If $a,b,c\in\mathbb{Z}$, then inequality (4) can be written as $\max(|a|,|b|,|c|)\le K_\varepsilon(\mathrm{rad}(abc))^{1+\varepsilon}$ [Oe]. The commonality of this inequality with inequality (4) is obvious, if $a,b,c\in\mathbb{N}$. Further, in inequality (4) always $K_\varepsilon<1$, in the high limit of $K_\varepsilon=1$ (5), which follows from (2). It should be noted, that condition (1) does not exclude hit triples is a,b,c, satisfying the inequality $c>\mathrm{rad}(abc)$. Herewith, $\varepsilon\ge 1$ is a very inflated estimate. It is quite enough, that $\varepsilon=\frac{2}{3}$ (6) [SY]. In other words, it is enough for him to be in the interval $\frac{2}{3}\le\varepsilon<1$, to always satisfy the inequality (4). Taking into account (5) inequality (4) takes the following form: $c<(\mathrm{rad}(abc))^{1+\varepsilon}$. If taking into account (6), then executed $c<(\mathrm{rad}(abc))^{1+\frac{2}{3}}$ (**). Let's clarify the formulation of the abc conjecture. For ε, such that $\frac{2}{3}\le\varepsilon<1$, exists K_ε, such that $0<K_\varepsilon\le 1$, at which for any three coprime

and ascending ordered numbers $a,b,c \in \mathbb{N}$, satisfying the equation $a+b=c$, the inequality $c \le K_\varepsilon(\text{rad}(abc))^{1+\varepsilon}$ holds $\otimes$.

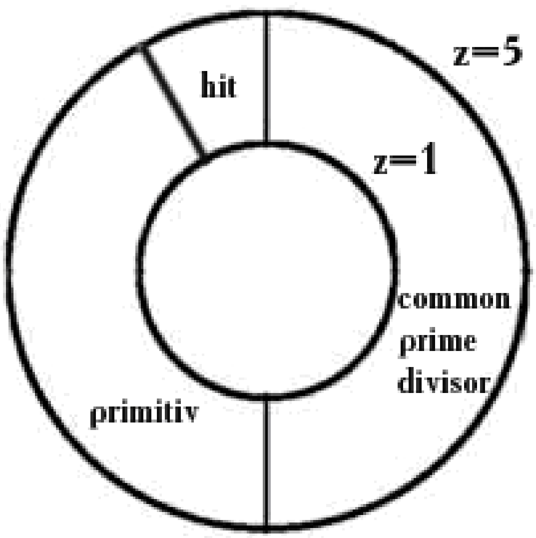
DETERMINATION. In equation (3) a,b,c can be either primitive or hit, the third is not given (*tertium non datur*) [Kol]. In equation (3) a,b,c these are always coprime numbers. Otherwise a,b,c have a common prime divisor and by definition are not coprime numbers (7). In other words, if in the equation (*) a,b,c are coprime numbers, then abc triples are primitive or hit. In any finite closed interval the number of any solutions (a,b,c) of equation (3), satisfying (7), is finite. If the solution (a,b,c) of equation (3) is hit, then a is any natural number, b and c are always composite (hit) numbers (8). Composite (hit) numbers are natural numbers that, can be represented as the product of any canonical prime factors [KR]. The Pythagorean triples of the numbers a,b,c in equation (3) are not hit (9).

z LEMMA. In any equation of the form (3), where a,b,c are represented as $a=A^x$, $b=B^y$, $c=C^z$ (10), z is limited (11).

PROOF. Suppose that an arbitrary equation (*) is given, the elements of which are a abc triple, satisfying (3) and (10). Then for him the following chain of inequalities follows from inequality (**) taking into account (5):

$$C^z < (\text{rad}(A^x B^y C^z))^{1+\frac{2}{3}} = (\text{rad}(ABC))^{1+\frac{2}{3}} < (ABC)^{1+\frac{2}{3}} < C^{3\times(1+\frac{2}{3})} = C^{3+2} = C^5. \quad (12)$$

It follows from (12), that $z<5$ (13). This means, that z it is limited (14) in the interval $1<z<5$ (15), which does not contradict (11) $\otimes$.



HIT CONJECTURES. The Catalan's conjecture [Ca] (Theorem Mihailescu's [Mi]) states that in the interval (15) the equation:

$$\underbrace{1}_{a}+\underbrace{B^{y}}_{b}=\underbrace{C^{z}}_{c} \tag{16}$$

if $B,C,y,z>1$ (is sufficient in the interval (15), then $z=2$ this is the lower bound z) has the only one solution in natural numbers $(B,C,y,z)=(2,3,3,2)\otimes$. In other words, there are no other consecutive perfect powers of natural numbers except $2^3=8$ and $3^2=9$, which follows from the list of known first hit triples (*sic erat scriptum*).

In Pillai's conjecture [Pi] it is stated that the equation:

$$\underbrace{A}_{a}+\underbrace{mB^{y}}_{b}=\underbrace{nC^{z}}_{c} \tag{17}$$

when $(y,z)\neq(2,2)$ (in the interval (15) it is sufficient, that $y,z>2$ and $z=2$ this is the lower bound z) (18) has a finite number of solutions (A,B,C,m,n,y,z) in natural numbers (19). In other words, the number of hit solutions of equation (17) in the interval (15) is finite. If in equation (17) $A,m,n=1$, then we have equation (16), that is the Catalan's conjecture is a special case of Pillai's conjecture. Therefore, one can follow from the particular to the general in order to prove Pillai conjecture [Pe], [So].

PROOF. Suppose (19) is incorrect. Then there are infinitely many solutions to equation (17) (20). It follows from the form of equation (16), that a this is any natural number, b and c these are composite (hit) numbers (21), which does not contradict (8). In the finite closed interval (15) there is a unique solution of equation (16) taking into account (10). This $(A,B,C,y,z)=(1,2,3,3,2)$. The corresponding abc triple $(a,b,c)=(1,8,9)$ is a hit. Since this solution is the only one, it can be argued, that the number of hit solutions of equation (16) in the interval (15) is finite (22). In any finite interval, as long as it is not less than the interval (15), (22) is true. Consequently, the solution of equation (16) satisfies (21) and (22). Let's make the transition from equation (16) to equation (17): $1\Rightarrow A$, $B^{y}\Rightarrow mB^{y}$, $C^{z}\Rightarrow nC^{z}$. It is known, that the triples of numbers $a,b,c\in\mathbb{N}$

(10) are either primitive (23) or hit (24). Otherwise a,b,c have a common prime divisor (25). Condition (18) excludes Pythagorean triples, that is excludes (23) and (25), satisfying (9) and leaving only hit solutions of equation (17) in the interval (15). It follows from this, that the number of any solutions of this equation in the interval (15) is limited, because they are all hit only, that is they satisfy (21). Then in any finite interval, as long as it is not less than interval (15) and satisfies condition (18), the number of hit solutions of equation (17) is finite (26). Since (26) contradicts (20), then (19) is true $\otimes$.

CONSEQUENCE. If in equation (17) $A,m,n=1$, $y,z>1$ (and $z=2$ this is the lower bound z), then if condition (18) is met, this equation has only a finite number of hit solutions in natural numbers (Tiedemann's Theorem) [Ti]. In other words, there are at most a finite number of consecutive power y,z of numbers B,C respectively, satisfying equation (17)$\otimes$. Since Pillai's conjecture is proved, Tiedemann's Theorem is redundant.

CONCLUSION. All hit conjectures in the interval (15) have a lower bound $z=2$.

It seems to us that this conclusion can be extended to any noncontroversial statement regarding the solvability of equation (*), containing a limitation z in the interval (15). However, here already *plus est dictum, quam necesse est.*

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