The inhomogeneity of a vector field as a sum of biquaternions, rotations, and spinors in a generalized Clifford algebra

Babaev A. Kh.

Abstract: In 4-dimensional curved space, the article presents the relations between the vector field inhomogeneity, biquaternions, rotations, and spinors. As a mathematical tool, the generalized Clifford algebra has been employed. The electromagnetic field inhomogeneity is proven to be made up of three independent rotations, biquaternions, and three pairs of spinors-antispinors.

Keywords: Biquaternion, bispinor, rotation in 4 - complex space, Clifford algebra, inhomogeneity.

1 Introduction

In an inhomogeneous space (field), combining rotations, biquaternions, and spinors within the extended Clifford algebra (field) affords universal mathematical tools for uniting the equations of Einstein, Maxwell, and Dirac. Bivectors, rotations in four-dimensional space, biquaternions, and bispinors are all repercussions of the vector field's local inhomogeneity.

1.1 Theoretical basis

The measure of local inhomogeneity of a vector field A was given in the paper [1]:

$$B = \nabla A \tag{1}$$

here $\nabla \equiv e^i D/\partial q^i \equiv e^i D_i$ is an operator nabla; e_i are vectors (4x4 matrices) of the base frame; $D/\partial q^i$ is a covariant derivative by to argument q^i.

According to the vector product of Clifford [2]:

$$B = \nabla \bullet A + \nabla \wedge A \tag{2}$$

here $\nabla \bullet A$ is an inner product of vectors; $\nabla \wedge A$ is an outer product of vectors.

$\nabla \bullet A$ and $\nabla \wedge A$ are identified with the deformation and rotation of the vector field A in the paper [1].

2 Results and Discussion

2.1 Rotations in a 4-dimensional space

Theorem: The local inhomogeneity $\boldsymbol{B}$ of the vector field (1) consists of the sum of independent rotations in 4-dimensional space:

$$\boldsymbol{B} = \sum_{\alpha=1}^{3} \left(|\boldsymbol{e}_\alpha \wedge \boldsymbol{e}_0| \cosh \frac{\boldsymbol{z}_\alpha}{2} + \boldsymbol{e}_\alpha \wedge \boldsymbol{e}_0 \sinh \frac{\boldsymbol{z}_\alpha}{2} \right) \tag{3}$$

here $|e_\alpha \wedge e_0| = |\tau_{\alpha 0}| = \sqrt{g_{\alpha 0} g_{\alpha 0} - g_{00} g_{\alpha\alpha}}$; $\tau_{\alpha 0} = e_\alpha \wedge e_0$; g_{ij} – is the metric tensor; $z_\alpha = \eta_\alpha + \gamma\, \varphi_\alpha$; η_α – is the rapidity or the hyperbolic angle of rotation on the hyperplane of axes α and 0 (q^α and q^0); φ_α – is the angle of a usual rotation around the axis q^α; $\gamma = \gamma_0\gamma_1\gamma_2\gamma_3$, γ_i – are the Dirac matrices.

Proof. Taking into account $F = \nabla \wedge A$, where F is the electromagnetic field tensor, we write equation (2) in the form:

$$B = \nabla \bullet A + F \tag{4}$$

We will separate $\boldsymbol{F}$ to electric and magnetic field:

$$\nabla \wedge A = F = e^i \wedge e^j F_{ij} = e^\alpha \wedge e^0 F_{\alpha 0} + e^\lambda \wedge e^\mu F_{\lambda\mu}$$

or

$$F = e^\alpha \wedge e^0 F_{\alpha 0} + \gamma\, (e_\alpha \wedge e_0)\, E^{\alpha 0 \lambda\mu} F_{\lambda\mu} = E + \gamma\, H \tag{5}$$

here $E^{\alpha 0\lambda\mu}$ is the Levi-Civita tensor of the fourth rank in the contravariant form; $E = e^\alpha \wedge e^0 F_{\alpha 0}$ is the electric field; $H = (e_\alpha \wedge e_0) E^{\beta\lambda\alpha 0} F_{\beta\lambda}$ is the magnetic field.

By putting (5) into (4) and denoting $\nabla \bullet A = S\, I$ (S is a scalar), we get:

$$B = S\, I + E + \gamma\, H \tag{6}$$

Squaring equation (6) and simplifying, we get:

$$B^2 = (S^2 + E^2 - H^2)\, I + \gamma\, E \bullet H + 2\, S\, E + 2\, \gamma\, SH \tag{7}$$

We denote the scalar, pseudoscalar, vector and pseudovector parts of the equation (7) as:

$SR = (S^2 + E^2 - H^2)\, I$ is a scalar; $SP = \gamma\, E \bullet H$ is a pseudoscalar;

$VR = 2\, S\, E$ is a vector; $VP = 2\, \gamma\, SH$ is a pseudovector.

Summing ***SR*** and ***SP*** and simplifying, we get the biscalar:

$$SR + SP = |\tau_{\alpha 0}||\tau_{\beta 0}| cosh(z_\alpha/2 + z_\beta/2) \tag{8}$$

Now summing ***VR*** and ***VP*** and simplifying, we get the bivector:

$$VR + VP = 2\, \tau_{\alpha 0}|\tau_{\beta 0}| sinh(z_\alpha/2)\, cosh(z_\beta/2) \tag{9}$$

Finally, summing up (8) and (9) and simplifying, we get:

$$B^2 = (|\tau_{\alpha 0}| cosh(z_\alpha/2) + \tau_{\alpha 0}\, sinh(z_\alpha/2))^2 \tag{10}$$

Taking the square root of (10) and summing over α from 1 to 3, we get the equation (3). The theorem has been proven. **Appendix 1** contains the computations in detail.

So, the local homogeneity ***B*** of a vector field ***A*** consists of three rotations in 4-dimensional space.

Expression (3) is the rotations to the complex angle $z_\alpha/2$ on the affine plane of the axes q^α, q^0. Swapping the indices α and 0, also λ and μ in the $e^\alpha \wedge e^0 F_{\alpha 0} + e^\lambda \wedge e^\mu F_{\lambda\mu}$, we get the reverse rotations identical to (3). Rotations (3) are essential because Lorentz transformations (including conventional rotations in 3-dimensional space) and spinors in a generalized form, i.e. in curved space, can be easily obtained from them.

Note. In the case of the Minkowski space ($g_{00} = 1$; $g_{\alpha\alpha} = -1$), these rotations are the Lorentz group ***SO(1,3)*** [3].

2.2 Bispinors

Now we get spinors from rotations (3). According to Euler's formula

$$cosh(z_\alpha/2) = 0.5(exp(z_\alpha/2) + exp(-z_\alpha/2)) = Y_\alpha + \bar{Y}_\alpha$$
$$sinh(z_\alpha/2) = 0.5(exp(z_\alpha/2) - exp(-z_\alpha/2)) = Y_\alpha - \bar{Y}_\alpha,$$

here

$$Y_\alpha = 0.5\, exp(z_\alpha/2);\quad \bar{Y}_\alpha = 0.5\, exp(-z_\alpha/2), \tag{11}$$

we can write the formula (3) in form:

$$B = \Sigma_\alpha(|\tau_{\alpha 0}| + \tau_{\alpha 0}) Y_\alpha + \Sigma_\alpha(|\tau_{\alpha 0}| - \tau_{\alpha 0}) \bar{Y}_\alpha \quad \boldsymbol{\alpha}=1,2,3. \tag{12}$$

Let's introduce the notation:

$$S_\alpha = (|\tau_{\alpha 0}| + \tau_{\alpha 0}) Y_\alpha \tag{13}$$

$$\check{S}_\alpha = (|\tau_{\alpha 0}| - \tau_{\alpha 0}) \bar{Y}_\alpha \tag{14}$$

Following the group theory's terminology, in the general case, we say that the ideal of a ring K is such a subring k for ∀b∈ K and ∀***S***∈ k that the following equality holds [4]:

$$\boldsymbol{Sb = cS}$$

where c – is a real number. If c>0 then ***S*** is a positive ideal (ideal), if c<0 then ***S*** is a negative ideal (anti-ideal). The term "anti-ideal" was introduced for the generality of concepts.

The ideals can be right and/or left. If an ideal is both left and right, then such an ideal is called a two-sided ideal or simply an ideal.

In 4-dimensional physical space, such ideals correspond to spinors [5]. We will check are there such ideals (spinors) in our case:

1. For $\boldsymbol{S_\alpha}$ (13):

$$S_\alpha (\tau_{\alpha 0}) =(|\tau_{\alpha 0}|+ \tau_{\alpha 0})Y_\alpha(e_\alpha \wedge e_0) = (|\tau_{\alpha 0}|(\tau_{\alpha 0}) + (\tau_{\alpha 0})(\tau_{\alpha 0}))Y_\alpha = (|\tau_{\alpha 0}|(\tau_{\alpha 0}) + (\tau_{\alpha 0})^2)Y_\alpha =|\tau_{\alpha 0}|(\tau_{\alpha 0} + |\tau_{\alpha 0}|)Y_\alpha = |\tau_{\alpha 0}|S_\alpha$$

So $\quad S_\alpha (e_\alpha \wedge e_0) =|e_\alpha \wedge e_0|S_\alpha$

2. For $\boldsymbol{\check{S}_\alpha}$ (14) in the same way we get:

$$\check{S}_\alpha (e_\alpha \wedge e_0) = - |e_\alpha \wedge e_0|\check{S}_\alpha$$

$\boldsymbol{S_\alpha}$ are called positive spinors (spinors) and are defined by formula (13);

$\boldsymbol{\check{S}_\alpha}$ is called negative spinors (antispinors) and is defined by formula (14).

Then the local inhomogeneity of the electromagnetic field (12) can be written as the sum of three spinors - antispinors pairs:

$$B = \Sigma_\alpha(S_\alpha + \check{S}_\alpha) \quad \boldsymbol{\alpha}=1,2,3. \tag{15}$$

Theorem: The ideals of positive and negative spinors are independent, i.e.

$$\Sigma_\alpha(n_\alpha S_\alpha + \check{n}_\alpha\check{S}_\alpha) = 0 \quad \boldsymbol{\alpha}=1,2,3. \tag{16}$$

i.e., the condition (16) is satisfied only if all real numbers $\boldsymbol{n_\alpha, \check{n}_\alpha}$ are simultaneously equal to zero (under the condition $\boldsymbol{S_\alpha \neq 0, \check{S}_\alpha \neq 0}$).

The proof of the independence of spinors is given in **Appendix 2**.

Below we present several consequences arising from the properties of spinors.

1. The ideals of spinors and antispinors are double-sided.

2. If a nonzero biquaternion $\boldsymbol{B}$ is a sum of spinors, then it satisfies the condition [6]:

$$S_\alpha\check{S}_\alpha = 0$$

Really $\quad S_\alpha\check{S}_\alpha = (|\tau_{\alpha 0}|+ \tau_{\alpha 0})Y_\alpha (|\tau_{\alpha 0}| - \tau_{\alpha 0})\bar{Y}_\alpha = (|\tau_{\alpha 0}|+ \tau_{\alpha 0})(|\tau_{\alpha 0}| - \tau_{\alpha 0})\, Y_\alpha\bar{Y}_\alpha =$

$$= (|\tau_{\alpha 0}||\tau_{\alpha 0}|+ \tau_{\alpha 0}|\tau_{\alpha 0}| - |\tau_{\alpha 0}|\, \tau_{\alpha 0} - \tau_{\alpha 0}\cdot\tau_{\alpha 0}- \tau_{\alpha 0}\wedge\tau_{\alpha 0})Y\alpha\bar{Y}\alpha =0\cdot 1=0$$

Spinors are crucial. We may get three pairs of Dirac equations by calculating the gradient from equation (15):

$$\Sigma_\alpha(\nabla S_\alpha +\nabla\, \check{S}_\alpha) = \Sigma_\alpha\nabla B_\alpha \quad \boldsymbol{\alpha}=1,2,3. \tag{17}$$

2.3 Biquaternions

There is a lot of literature about quaternions, and their relationship with rotations and spinors [7,8]. Therefore, we confine ourselves to describing biquaternions in the generalized case.

Expanding $\cosh((\eta_\alpha + \gamma\, \varphi_\alpha) /2)$ and $\sinh((\eta_\alpha + \gamma\, \varphi_\alpha) /2)$ to the sum of arguments (in formula (3)) and separating into scalar, pseudoscalar, bivector and pseudobivector parts, we get a biquaternions in generalized form:

$$B_\alpha = I\, s_\alpha + \tau_{\alpha 0}\, b_\alpha + \gamma\, ps_\alpha + \gamma\, \tau_{\alpha 0}\, pb_\alpha \tag{18}$$

here $I\, s_\alpha$ – is a scalar, $\gamma\, ps_\alpha$ – is a pseudoscalar, $e_\alpha \wedge e_0\, b_\alpha$ – is a bivector, $\gamma\, e_\alpha \wedge e_0\, pb_\alpha$ – is a pseudobivector. $\alpha = 1,2,3.$

It can be seen from (18) that the local inhomogeneity of the vector field ***B*** with potential ***A*** consists of the sum of three real quaternions $I\, s_\alpha + e_\alpha \wedge e_0\, b_\alpha$ and three imaginary quaternions $\gamma\, ps_\alpha + \gamma\, e_\alpha \wedge e_0\, pb_\alpha$.

3 Conclusions

1. The total of three independent biquaternions, rotations on the plane q^α, q^0 (Lorentz transformations) + ordinary rotations in 3-dimensional space, and three spinors-antispinors pairs in generalized form makes up the local inhomogeneity of the electromagnetic field.

2. Perhaps the presence of three pairs of independent biquaternions, accordingly, three pairs of spinor - antispinor are the reason for the existence of only three generations of leptons and quarks in four -dimensional space.

Appendix 1

We sum up **SR** and **SP**, then **VR** and **VP** from equation (7):

$$SR+SP=|\tau_{\alpha 0}||\tau_{\beta 0}|\cosh((\eta_\alpha+\eta_\beta)/2+\gamma(\varphi_\alpha+\varphi_\beta)/2)=|\tau_{\alpha 0}||\tau_{\beta 0}|\cosh(z_\alpha/2+z_\beta/2) \tag{19}$$

$$VR+VP = \tau_{\alpha 0}|\tau_{\beta 0}|(\sinh((\eta_\alpha+\eta_\beta)/2+\gamma(\varphi_\alpha+\varphi_\beta)/2) - \sinh((\eta_\alpha - \eta_\beta +\gamma\varphi_\alpha- \gamma\varphi_\beta)/2))$$

$$VR +VP = \tau_{\alpha 0}|\tau_{\beta 0}|(\sinh((z_\alpha+ z_\beta)/2) - \sinh((z_\alpha - z_\beta)/2))$$

$$VR +VP =2\, \tau_{\alpha 0}|\tau_{\beta 0}|\sinh(z_\alpha /2) \cosh(z_\beta /2) \tag{20}$$

Note that on the right side of equations (19) and (20) there is a summation over **α** and **β** from 1 to 3.

Summing up (19) and (20), we get:

$$B^2 = |\tau_{\alpha 0}||\tau_{\beta 0}|\cosh(z_\alpha/2+ z_\beta/2) +2\, \tau_{\alpha 0}|\tau_{\beta 0}|\sinh(z_\alpha/2) \cosh(z_\beta/2)$$

$$B^2 = |\tau_{\alpha 0}||\tau_{\beta 0}|(\cosh(z_\alpha/2)\cosh(z_\beta/2) + \sinh(z_\alpha/2)\sinh(z_\beta/2)) +2\tau_{\alpha 0}|\tau_{\beta 0}|\sinh(z_\alpha/2) \cosh(z_\beta/2)$$

$$B^2 = (|\tau_{\alpha 0}|\cosh(z_\alpha/2)+\tau_{\alpha 0} \sinh(z_\alpha/2))\ (|\tau_{\beta 0}|\cosh(z_\beta/2) +\tau_{\beta 0} \sinh(z_\beta/2))$$

Since **α** and **β** are summed from 1 to 3, we can write this equation in the form (10). Extracting from the (10), we get (3).

Appendix 2

The proof of the independence of spinors and antispinors:

We multiply the equation (16) to $(|\tau_{10}| - \tau_{10})$ from left side:

$$(|\tau_{10}| - \tau_{10})n_1S_1 + (|\tau_{10}| - \tau_{10})n_2S_2 + (|\tau_{10}| - \tau_{10})n_3S_3 +(|\tau_{10}| - \tau_{10})\, \Sigma_\alpha\, \check{n}_\alpha\check{S}_\alpha =0 \tag{21}$$

Since $(|\tau_{10}| - \tau_{10})n_1S_1 = n_1(|\tau_{10}| - \tau_{10})(|\tau_{10}| - \tau_{10})Y_1=0$, then we get from (21):

$$n_2S_2 + n_3S_3 +\Sigma_\alpha\, \check{n}_\alpha\check{S}_\alpha = 0 \tag{22}$$

So we multiply the equation (22) to $(|e_2\wedge e_0| - e_2\wedge e_0)$:

$$(|\tau_{20}| - \tau_{20})n_2S_2 +(|\tau_{20}| - \tau_{20})n_3S_3 +(|\tau_{20}| - \tau_{20})\Sigma_\alpha\, \check{n}_\alpha\check{S}_\alpha =0 \tag{23}$$

Here $(|e_2\wedge e_0| - e_2\wedge e_0)n_2S_2 = 0$, therefor from (23) we get:

$$n_3S_3 +\Sigma_\alpha\, \check{n}_\alpha\check{S}_\alpha = 0 \tag{24}$$

Further, repeating the multiplication (24) to ($|e_3\wedge e_0|$ - $e_3\wedge e_0$), then to ($|e_1\wedge e_0|$+ $e_1\wedge e_0$), etc. and repeating the procedure, at the end, we get:

$$(|\tau_{30}|+ \tau_{30})\check{n}_3\, \check{S}_3 = \check{n}_3(|\tau_{30}|+ \tau_{30})(|\tau_{30}|- \tau_{30})\bar{Y}_3 = 0 \tag{25}$$

Since neither ($|e_3\wedge e_0|$+ $e_3\wedge e_0$) nor $\mathbf{\check{S}_3}$ are equal to zero in (25), it follows that $\mathbf{\check{n}_3} = 0$.

Now we write equation (16) without the term $\check{n}_3\, \check{S}_3$:

$$\Sigma_\alpha\, n_\alpha\, S_\alpha + \check{n}_1\, \check{S}_1 + \check{n}_2\, \check{S}_2 = 0. \tag{26}$$

Repeating the procedure (22) - (25) with respect to equation (26), we finally get:

$$\check{n}_2\, (|e_2\wedge e_0|+ e_2\wedge e_0)\, (|e_2\wedge e_0|- e_2\wedge e_0)\bar{Y}_3 =0, \tag{27}$$

i.e. $\check{n}_2 =0$. Repeating these steps we can finally verify that all $n_\alpha = \check{n}_\alpha =0$.

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Note This paper is a partial translation and a more mathematically general and strictly proven version of the author's article in Russian, published in the reviewed journal: https://sci-article.ru/stat.php?i=1495946354