Line integrals in the concept of hypercomplex numbers within Clifford algebra

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Abstract: The paper presents the relation between line and surface integrals in Clifford algebra ($\mathscr{E}_4$) and, in particular, in Cartesian space ($\mathscr{E}_3$). The bijection between hypercomplex numbers and elements of space $\mathscr{E}_4$, in particular $\mathscr{E}_3$, has been set. The generalized Stokes theorem and Cauchy's integral theorem are generalized and combined into one. The physical interpretation of the formulas is in accord with the laws of the circulation of the electromagnetic field and gives some nontrivial results.

Keywords: Hypercomplex number, line integral, surface integral, Cauchy's integral formula, Stokes theorem, Dirac matrices, Pauli matrices, Clifford algebra.

Introduction

The bijection between vectors and complex numbers is obvious from the theory of functions of a complex argument in two-dimensional flat space [**1**]. But this doesn't seem obvious in a many-dimensional space (d > 2). We shall search for the relation between line and surface integrals in Minkowski space and three-dimensional Euclidean space in Clifford's algebra in this article. Bijection between hypercomplex numbers and elements of Clifford's algebra (spaces $\mathscr{E}_4$ and $\mathscr{E}_3$) makes it possible to generalize linear and superficial integrals in a hypercomplex space. We will use the Pauli matrices $\sigma_\alpha(\alpha=1,2,3)$ (because the space signature is +++) as basis vectors in the case of a 3-dimensional space and the Dirac matrices $\gamma_i(i=0,1,2,3)$ for the Minkowski space (-+++).

Results

I. *Spatial case (d=3)*

Let a positively oriented surface D with contour l be given in the space XYZ (Figure 1). The normal $\boldsymbol{n}$ makes an angle α, β, γ with the coordinate axes x, y, z.

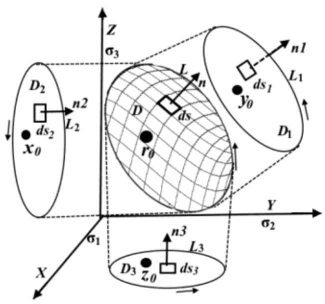

Fig. 1

Let a vector-function $\boldsymbol{A}(x,y,z) = \sigma_1 A_1(x,y,z) + \sigma_2(x,y,z) + \sigma_3 A_3(x,y,z)$ be given in a domain D.

We will consider the integrals of $\oint_l \boldsymbol{A}\, d\boldsymbol{l}$ and $\iint_D (\boldsymbol{\nabla A}) d\boldsymbol{s}$ in 3-dimensional Euclidean space.

here $d\boldsymbol{l} = \sigma_1 dx + \sigma_2 dy + \sigma_3 dz$ is the elementary arc length (vector);

$d\boldsymbol{s} = \boldsymbol{n}ds = (\sigma_1\sigma_2 cos\gamma + \sigma_2\sigma_3 cos\alpha + \sigma_3\sigma_1 cos\beta)ds$ is the surface element;

$\sigma_2\sigma_3 cos\alpha\ ds = i\sigma_1 dydz;\quad \sigma_3\sigma_1 cos\beta\ ds = i\sigma_2 dzdx;\quad \sigma_1\sigma_2 cos\gamma\ ds = i\sigma_3 dxdy;$

$\boldsymbol{\nabla} \equiv \sigma_1\partial_x + \sigma_2\partial_y + \sigma_3\partial_z$ is the nabla symbol;

$\boldsymbol{n} = \sigma_1\sigma_2 cos\gamma + \sigma_2\sigma_3 cos\alpha + \sigma_3\sigma_1 cos\beta$ is a positively oriented normal to the surface D.

According to Clifford product of vectors [**2**]

$$\oint_l \boldsymbol{A}\, d\boldsymbol{l} = \oint_l \boldsymbol{A} \bullet d\boldsymbol{l} + \oint_l \boldsymbol{A} \wedge d\boldsymbol{l} \quad \textbf{(1)}$$

$$\iint_D (\boldsymbol{\nabla A})\mathrm{d}\boldsymbol{s} = \iint_D (\boldsymbol{\nabla} \bullet \boldsymbol{A})\mathrm{d}\boldsymbol{s} + \iint_D (\boldsymbol{\nabla} \wedge \boldsymbol{A}) \bullet \mathrm{d}\boldsymbol{s} + \iint_D (\boldsymbol{\nabla} \wedge \boldsymbol{A}) \wedge \mathrm{d}\boldsymbol{s} \quad \textbf{(2)}$$

We will prove some complex analysis theorems for three-dimensional space.

Theorem 1. The following formula is correct:

$$\oint_l \boldsymbol{A}\, d\boldsymbol{l} = \iint_D (\boldsymbol{\nabla A})\mathrm{d}\boldsymbol{s} \quad \textbf{(3)}$$

or

$$\oint_l \boldsymbol{A} \bullet d\boldsymbol{l} = \iint_D (\boldsymbol{\nabla} \wedge \boldsymbol{A}) \bullet \mathrm{d}\boldsymbol{s} \quad \textbf{(4)}$$

$$\oint_l \boldsymbol{A} \wedge d\boldsymbol{l} = \iint_D (\boldsymbol{\nabla} \bullet \boldsymbol{A})\mathrm{d}\boldsymbol{s} \quad \textbf{(5)}$$

$$\iint_D (\boldsymbol{\nabla} \wedge \boldsymbol{A}) \wedge \mathrm{d}\boldsymbol{s} = \boldsymbol{0} \quad \textbf{(6)}$$

The proof of Theorem 1 is given in Appendix 1.

So we get the generalized Stokes formula (3), or (4), (5), (6).

When integrating, we assumed that the domain D is simply connected and that the function itself has no singularities in the domain D.

We will not consider splitting a non-simply-connected domain into simply-connected domains, since this procedure is sufficiently described in the classical literature [**4**].

Now we consider the case when the function has singularities in the integration domain.

Theorem 2

Let the function $\boldsymbol{A}(x,y,z)$ with its first derivatives be defined at all points in the domain *D*. In other words, the function is analytic in the domain *D*. Then the following formula is correct:

$$\oint_l \boldsymbol{A}\, d\boldsymbol{l} = 0, \qquad \textbf{(7)}$$

i.e., if the function $\boldsymbol{A}(x,y,z)$ is analytic in the domain *D*, then the closed-loop integral *l* in this domain *D* is equal to zero. Conversely, if the line integral on the closed-loop *l* is equal to zero (7), then the function is analytic in the domain *D*.

Proof. We will expand the integral $\oint_l \boldsymbol{A}\, d\boldsymbol{l}$ over the surfaces *X0Y*, *Y0Z*, and *Z0X*. For example, on the *X0Y* plane, this integral has the form:

$$\oint_{l3} \boldsymbol{A}\, d\boldsymbol{l} = \oint_{l3} (\sigma_1 A_1 + \sigma_2 A_2)(\sigma_1 dx + \sigma_2 dy)$$

We transform the integrand

$$\oint_{l3} \boldsymbol{A}\, d\boldsymbol{l} = \oint_{l3} (\sigma_0 A_1 + \sigma_2\sigma_1 A_2)(\sigma_0 dx + \sigma_1\sigma_2 dy) = \oint_{l3} (\sigma_0 A_1 - i\sigma_3 A_2)(\sigma_0 dx + i\sigma_3 dy)$$

Since the function $\boldsymbol{A}(z)=\sigma_0 A_1 + i\sigma_3 A_2$ ($z=\sigma_0 x + i\sigma_3 y$) is analytic, i.e., the Cauchy-Riemann condition is satisfied (when changing $A_2 \rightarrow -A_2$):

$$\frac{\partial A_1}{\partial x} = \frac{\partial A_2}{\partial y}, \quad \frac{\partial A_1}{\partial y} = -\frac{\partial A_2}{\partial x},$$

then the integral on the closed-loop l_3 is equal to 0. The proofs are similar for the planes *Y0Z* and *Z0X*. Theorem 2 (formula (7)) is proved.

Now, we assume that in the *D* domain the $\boldsymbol{A}(\boldsymbol{r})$ function has a simple pole i.e. a singularity of the form:

$$\boldsymbol{A}(r) = \frac{f(r)}{\boldsymbol{r}-\boldsymbol{r}_0} \qquad \textbf{(8)}$$

Theorem 3

If the function $\boldsymbol{A}(\boldsymbol{r})$ is analytic in the domain *D*, except at the point r0, and has a simple pole (8) in this point, then the following formula is correct:

$$\oint_l \frac{f(r)}{\boldsymbol{r}-\boldsymbol{r_0}} d\boldsymbol{l} = 2\, i\, \pi\, \boldsymbol{n}\, f(\boldsymbol{r_0}) \qquad \textbf{(9)}$$

here $\boldsymbol{n}$ is normal to the surface.

Proof. We'll change the integral.

$$\oint_l \frac{f(\boldsymbol{r})d\boldsymbol{l}}{\boldsymbol{r}-\boldsymbol{r_0}} = \oint_l \frac{f(\boldsymbol{r})-f(\boldsymbol{r_0})+f(\boldsymbol{r_0})}{\boldsymbol{r}-\boldsymbol{r_0}} d\boldsymbol{l} = \oint_l \frac{f(\boldsymbol{r})-f(\boldsymbol{r_0})}{\boldsymbol{r}-\boldsymbol{r_0}} d\boldsymbol{l} + f(\boldsymbol{r_0}) \oint_l \frac{d\boldsymbol{l}}{\boldsymbol{r}-\boldsymbol{r_0}}$$

$$\frac{f(\boldsymbol{r})-f(\boldsymbol{r_0})}{\boldsymbol{r}-\boldsymbol{r_0}} \Rightarrow f'(\boldsymbol{r}) \quad \text{at } \boldsymbol{r} \to \boldsymbol{r}_0.$$

Of course, the first integral is equal to 0:

$$\oint_l \frac{f(\boldsymbol{r})-f(\boldsymbol{r_0})}{\boldsymbol{r}-\boldsymbol{r_0}} d\boldsymbol{l} = \oint_{\boldsymbol{l}} f_r'(\boldsymbol{r}) d\boldsymbol{l} = f(\boldsymbol{r})|_l = 0$$

Projecting the second integral ($\oint_l \frac{d\boldsymbol{l}}{\boldsymbol{r}-\boldsymbol{r_0}}$) onto the X0Y, Y0Z, and Z0X planes and replacing $\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_0$, we get:

$$\oint_l \frac{d\boldsymbol{l}}{\boldsymbol{r}-\boldsymbol{r_0}} = \oint_l \frac{\sigma_1 dx+\sigma_2 dy+\sigma_3 dz}{\sigma_1 x+\sigma_2 y+\sigma_3 z} \Rightarrow \oint_{l3} \frac{\sigma_1 dx+\sigma_2 dy}{\sigma_1 x+\sigma_2 y} + \oint_{l2} \frac{\sigma_1 dx+\sigma_3 dz}{\sigma_1 x+\sigma_3 z} + \oint_{l1} \frac{\sigma_2 dy+\sigma_3 dz}{\sigma_2 y+\sigma_3 z}$$

For each integral, we apply the proof of Cauchy's integral formula [**5**].

For example, $\oint_{l3} \frac{\sigma_1 dx+\sigma_2 dy}{\sigma_1 x+\sigma_2 y} = \oint_{l3} \frac{dx+\sigma_1\sigma_2 dy}{x+\sigma_1\sigma_2 y} = \oint_{l3} \frac{dx+i\sigma_3 dy}{x+i\sigma_3 y} = i\sigma_3 \int_0^{2\pi} \frac{re^{i\sigma_3\varphi}}{re^{i\sigma_3\varphi}} d\varphi = 2\pi i\sigma_3$

In the calculation, we applied the parameterization $x + i\sigma_3 y = re^{i\sigma_3\varphi}$, $\varphi \in [0;2\pi]$.

After all, we have

$$f(\boldsymbol{r}_0) \oint_l \frac{d\boldsymbol{l}}{\boldsymbol{r}-\boldsymbol{r}_0} = 2\pi i\, f(\boldsymbol{r}_0)(\sigma_1 cos\alpha + \sigma_2 cos\beta + \sigma_3 cos\gamma) = 2\pi i\, \boldsymbol{n}\, f(\boldsymbol{r}_0)$$

Theorem 3 (formula (9)) is proved.

Formula (9) is a generalization of the integral formula of Cauchy for three-dimensional space.

Corollary

If the $\boldsymbol{A}(x, y, z)$ function has all the derivatives up to *k*-order and has the form $\frac{f(\boldsymbol{r})}{(\boldsymbol{r}-\boldsymbol{r_0})^{k+1}}$, then the following formula is correct:

$$\oint_l \frac{f(\boldsymbol{r})d\boldsymbol{l}}{(\boldsymbol{r}-\boldsymbol{r_0})^{k+1}} = 2\,\pi i\, \boldsymbol{n}\, k!\; f^{(k)}(\boldsymbol{r_0}) \tag{10}$$

By combining (6), (7), and (9), we can conclude:

1. If the $\boldsymbol{A}(x,y,z)$ function is analytic in the *D* domain, then the following formula is true:

$$\oint_l \boldsymbol{A}\, d\boldsymbol{l} = \oint_l \boldsymbol{A} \bullet d\boldsymbol{l} + \oint_l \boldsymbol{A} \wedge d\boldsymbol{l} = \iint_D (rot\boldsymbol{A}) \bullet d\boldsymbol{s} + i \iint_D (div\boldsymbol{A}) d\boldsymbol{s} = \boldsymbol{0} \tag{11}$$

Separating (11) into symmetric (real) and antisymmetric (imaginary) parts, we get the classical formulas:

$$\oint_l \boldsymbol{A} \bullet d\boldsymbol{l} = \iint_D (rot\boldsymbol{A}) \bullet d\boldsymbol{s} = \boldsymbol{0} \tag{11.a}$$

$$\oint_l \boldsymbol{A} \wedge d\boldsymbol{l} = \iint_D (\,div\boldsymbol{A}) d\boldsymbol{s} = \boldsymbol{0} \tag{11.b}$$

2. If the function has a simple pole, i.e. has the form $\boldsymbol{A}(\boldsymbol{r}) = \frac{f(\boldsymbol{r})}{\boldsymbol{r}-\boldsymbol{r}_0}$, then the following formula is true:

$$\oint_l \boldsymbol{A}\, d\boldsymbol{l} = \oint_l \boldsymbol{A} \bullet d\boldsymbol{l} + \oint_l \boldsymbol{A} \wedge d\boldsymbol{l} = \iint_D (\,rot\boldsymbol{A}) \bullet d\boldsymbol{s} + i \iint_D (\,div\boldsymbol{A}) d\boldsymbol{s} = 2\pi i f(\boldsymbol{r}_0) \tag{12}$$

Separating (12) into symmetric (real) and antisymmetric (imaginary) parts, we get the classical formulas:

$$\oint_l \boldsymbol{A} \bullet d\boldsymbol{l} = \iint_D (\,rot\boldsymbol{A}) \bullet d\boldsymbol{s} = 0 \tag{12.a}$$

$$\oint_l \boldsymbol{A} \wedge d\boldsymbol{l} = i \iint_D (\,div\boldsymbol{A}) d\boldsymbol{s} = 2\pi i f(\boldsymbol{r}_0) \tag{12.b}$$

Formula (12) unites the Generalized Stokes formula (11) with the Generalized Cauchy's integral formula.

II. *Space-time case* (*d*=*4 – Minkowski space*)

Now we consider the relationship between the integral line ($\oint_l A\, dl$) and the integral surface ($\iint_D (\nabla A) ds$) in the Minkowski space. As a basis, Dirac's matrices are used in the following representation:

$$\gamma^0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix}, \gamma^\alpha = \begin{bmatrix} 0 & i\sigma_\alpha \\ i\sigma_\alpha & 0 \end{bmatrix}.$$

where σ_i are Pauli matrices.

We consider the function $A = \gamma^i A_i(x_j)$. Leave a positively oriented hypersurface *D* with its boundary *l* is given in the *TXYZ* space.

Theorem 4

The following formula is correct

$$\oint_l A\, dl = \iint_D (\nabla A) ds \tag{13}$$

Here $dl = \gamma^i dx_i$; $\nabla = \gamma^j \partial_i$ is the nabla operator;

$dS=Nds$ is an element of the 4-dimensional hyperplane *S*;

$N=\gamma^0\gamma^1 cos\alpha_{01}+\gamma^0\gamma^2 cos\alpha_{02}+\gamma^0\gamma^3 cos\alpha_{03}+\gamma^2\gamma^1 cos\alpha_{12}+\gamma^1\gamma^3 cos\alpha_{13}+\gamma^3\gamma^2 cos\alpha_{23}$ is the normal to the surface *S*;

α_{01}, α_{02},… are the angles between *N* and the direction vectors of the hyperplanes *TX* ($\gamma^0\gamma^1$), *TY*($\gamma^0\gamma^2$), etc. In other words, $cos\alpha_{01}$, $cos\alpha_{02}$,… are direction cosines.

$x_0=t, x_1=x, x_2=y, x_3=z;$

$\gamma^0\gamma^1 cos\alpha_{01}ds=\gamma^0\gamma^1 dtdx;$ $\gamma^0\gamma^2 cos\alpha_{02}ds=\gamma^0\gamma^2 dtdy;$ $\gamma^0\gamma^3 cos\alpha_{03}ds=\gamma^0\gamma^3 dtdz;$ $\gamma^2\gamma^1 cos\alpha_{12}ds=\gamma^2\gamma^1 dxdy;$ $\gamma^1\gamma^3 cos\alpha_{13}ds=\gamma^1\gamma^3 dxdz;$ $\gamma^3\gamma^2 cos\alpha_{23}ds=\gamma^3\gamma^2 dydz;$

In Minkowski space, the line integral of the function A over the contour l is equal to the surface integral of ∇A over the domain D bounded by the contour l. Theorem 4 (formula (13)) is a specific case (4 dimensions) of a mixed Hodge structure [**6**].

Taking into account Clifford product, also $F=\nabla\wedge A$, equation (13) is written as:

$$\oint_l A \bullet dl + \oint_l A \wedge dl = \iint_D (\nabla \bullet A) dS + \iint_D F \bullet dS + \iint_D F \wedge dS \qquad \textbf{(14)}$$

Comparing the left and right sides of the equation (14) similar to (3), (4), (5) and (6), we get:

$$\oint_l A \bullet dl = \iint_D F \bullet dS \qquad \textbf{(14.1)}$$

$$\oint_l A \wedge dl = \iint_D (\nabla \bullet A) dS \qquad \textbf{(14.2)}$$

$$\iint_D F \wedge dS = 0 \qquad \textbf{(14.3)}$$

Note:

$$\gamma^0\gamma^1 = \begin{bmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{bmatrix}; \gamma^0\gamma^2 = \begin{bmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix}; \gamma^0\gamma^3 = \begin{bmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{bmatrix};$$

$$\gamma^2\gamma^1 = \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix}; \ \gamma^1\gamma^3 = \begin{bmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{bmatrix}; \ \gamma^3\gamma^2 = \begin{bmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{bmatrix}.$$

$$\gamma^0\gamma^0 = -\gamma^1\gamma^1 = -\gamma^2\gamma^2 = -\gamma^3\gamma^3 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix}.$$

The proof of Theorem 4 (formula (13)) is given in **Appendix** 2.

Now we will prove theorems for the Minkowski space, like a 3-dimensional space, thus generalizing them.

We skip to divide a non-simply connected domain into simple connected ones, since this method was described many things in classical literature.

Theorem 5

If the function $A=\gamma^i A_i(x_j)$ and its first derivatives are defined in all points of the domain D, i.e., if it is analytic in the region D, then the following formula is true:

$$\oint_l A\,dl = 0 \qquad \textbf{(15)}$$

If the function $A=\gamma^i A_i(x_j)$ is analytic in the domain D, then the line integral along the closed-loop l is equal to zero. And vice versa, if (15) is satisfied, then the function is analytic.

We present Theorem 5 without proof because it is similar to the three-dimensional case.

Now we consider the case where the $A=\gamma^i A_i(x_j)$ function in the D domain has a singular point (the four-dimensional pole):

$$A(\rho) = \frac{f(\rho)}{\rho-\rho_0}$$

where $\rho = \gamma^i x_i$ is a 4-dimensional radius vector or interval; $dl \cong d\rho$; f (ρ) is a scalar function. "Einstein notation" indices mean summation.

Theorem 6

Let the function $A(\rho)$ be analytic in the domain D except at the point ρ_0. Let the function A have the first-order pole at the point ρ_0, i.e., one has the form $\frac{f(\rho)}{\rho-\rho_0}$. Then the following formula is true:

$$\oint_l \frac{f(\rho)}{\rho-\rho_0} dl = 2\pi i\, N f(\rho_0) \qquad \textbf{(16)}$$

$N = \gamma^0\gamma^1 \cos\alpha_{01} + \gamma^0\gamma^2 \cos\alpha_{02} + \gamma^0\gamma^3 \cos\alpha_{03} + \gamma^2\gamma^1 \cos\alpha_{21} + \gamma^1\gamma^3 \cos\alpha_{13} + \gamma^3\gamma^2 \cos\alpha_{32}$ is the normal.

In other words, the formula (16) is a generalization of Cauchy's integral formula in the Minkowski space.

Proof. We transform the integral:

$$\oint_l \frac{f(\rho)}{\rho-\rho_0} dl = \oint_l \frac{f(\rho)-f(\rho_0)+f(\rho_0)}{\rho-\rho_0} dl = \oint_l \frac{f(\rho)-f(\rho_0)}{\rho-\rho_0} dl + \oint_l \frac{f(\rho_0)}{\rho-\rho_0} dl$$

The expression under the first integral is the derivative at $\rho \to \rho_0$. This integral is equal to zero:

$$\oint_l \lim_{\rho\to\rho_0} \frac{f(\rho)-f(\rho_0)}{\rho-\rho_0} dl = \oint_l f'(\rho)\,dl = f(\rho)|_l = 0$$

Since the function $f(\rho_0) = const$ in the point ρ_0, we will consider the integral $\oint_l \frac{dl}{\rho-\rho_0}$. We project this integral over hyperplanes and perform a parameterization ($t - t_0 = t, x - x_0 = x, y - y_0 = y, z - z_0$):

$$\oint_l \frac{dl}{\rho-\rho_0} = \oint_l \frac{\gamma^0 dt+\gamma^1 dx+\gamma^2 dy+\gamma^3 dz}{\gamma^0 t+\gamma^1 x+\gamma^2 y+\gamma^3 z} =$$

$$= \oint_{lT0X} \frac{\gamma^0 dt+\gamma^1 dx}{\gamma^0 t+\gamma^1 x} + \oint_{lT0Y} \frac{\gamma^0 dt+\gamma^2 dy}{\gamma^0 t+\gamma^2 y} + \oint_{lT0Z} \frac{\gamma^0 dt+\gamma^3 dz}{\gamma^0 t+\gamma^3 z} + \oint_{lX0Y} \frac{\gamma^1 dx+\gamma^2 dy}{\gamma^1 x+\gamma^2 y} + \oint_{lX0Z} \frac{\gamma^1 dx+\gamma^3 dz}{\gamma^1 x+\gamma^3 z} + \oint_{lY0Z} \frac{\gamma^2 dy+\gamma^3 dz}{\gamma^2 y+\gamma^3 z}$$

We consider each integral separately. For example, for the "purely spatial" plane *X0Y*:

$$\oint_{lX0Y} \frac{\gamma^1 dx+\gamma^2 dy}{\gamma^1 x+\gamma^2 y} = \oint_{lX0Y} \frac{\begin{bmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{bmatrix} dx + \begin{bmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} dy}{\begin{bmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} y} = \oint_{lX0Y} \frac{\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} dx + \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} dy}{\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} y} =$$

$$\equiv \oint_{lX0Y} \frac{\sigma_1 dx+\sigma_2 dy}{\sigma_1 x+\sigma_2 y} = \oint_{lX0Y} \frac{\sigma_0 dx+\sigma_1\sigma_2 dy}{\sigma_0 x+\sigma_1\sigma_2 y} = \oint_{lX0Y} \frac{\sigma_0 dx+i\sigma_3 dy}{\sigma_0 x+i\sigma_3 y} = i\sigma_3 \cos\alpha_{12} \int_0^{2\pi} \frac{re^{i\sigma_3\varphi}}{re^{i\sigma_3\varphi}} d\varphi = 2\pi i\sigma_3 \cos\alpha_{12}$$

For the planes *Y0Z* and *X0Z*, the proof is similar.

For "purely time" hyperplanes, for example, *T0Z*:

$$\oint_{lT0Z} \frac{\gamma^0 dt+\gamma^3 dz}{\gamma^0 t+\gamma^3 z} = \oint_{lT0Z} \frac{Idt+\gamma^0\gamma^3 dz}{It+\gamma^0\gamma^3 z} = \oint_{lT0Z} \frac{\begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix} dt + \begin{bmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{bmatrix} dz}{\begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix} t + \begin{bmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{bmatrix} z} \equiv \oint_{lT0Z} \frac{\sigma_0 dt \pm i\sigma_3 dz}{\sigma_0 t \pm i\sigma_3 z} = \pm i\sigma_3 \cos\alpha_{03} \int_0^{2\pi} \frac{\rho e^{\pm i\sigma_3\eta}}{\rho e^{\pm i\sigma_3\eta}} d\eta =$$

$$\pm 2\pi i\sigma_3 \cos\alpha_{03}$$

For the planes *T0X* and *T0Y*, the proof is similar.

Summing up all the integrals over the planes, we get the equation (16). Theorem 6 is proved.

Note.

Here η is the rapidity, $\tanh(\eta) = \frac{v}{c}$, v is the speed, c is the speed of light in vacuum. On the real (physical) plane $-\infty < \eta < +\infty$, but on the complex plane (ct, ix), η is the imaginary "rotation" angle – $i\eta$. Therefore, we can formally take the boundaries $0 < i\eta < 2\pi$, since $e^{\pm i\sigma_3\eta} = \sigma_0 \cos\eta \pm i\sigma_3 \sin\eta$. This is the main point of Dirac matrices acceptance in our representation.

Corollary

If the function *A*(*t*,*x*,*y*,*z*) is *k*-times differentiable and has the form $\frac{f(\rho)}{(\rho-\rho_0)^{k+1}}$, then the following formula is true:

$$\oint_l \frac{f(\rho)d\rho}{(\rho-\rho_0)^{k+1}} = 2\,\pi\,i\,N\,k!\;f^{(k)}(\rho_0) \qquad \textbf{(17)}$$

Generalizing Theorems 4, 5, and 6 (formulas (13), (14), (15), and (16)), we can conclude:

1. If the function $A=\gamma^i A_i(x_j)$ is analytic at all points of the domain *D*, then the following formula is true:

$$\oint_l A \bullet dl + \oint_l A \wedge dl = \iint_D (\nabla \bullet A) ds + \iint_D (\nabla \wedge A) \bullet ds + \iint_D (\nabla \wedge A) \wedge ds = 0 \qquad \textbf{(18)}$$

or

$$\oint_l A \bullet dl = \iint_D (\nabla \wedge A) \bullet ds = 0 \qquad \textbf{(19)}$$

$$\oint_l A \wedge dl = \iint_D (\nabla \bullet A) ds = 0 \quad \textbf{(20)}$$

$$\iint_D (\nabla \wedge A) \wedge ds = 0 \quad \textbf{(21)}$$

2. If the function $A=\gamma^i A_i(x_j)$ is analytic in the domain D, except at the point ρ, also one has a so-called 4-dimensional pole of the first order at this point r (it has the form $A(\rho) = f(\rho)/(\rho - \rho_0)$), then the following formula is true:

$$\oint_l A \bullet dl + \oint_l A \wedge dl = \iint_D (\nabla \bullet A) ds + \iint_D (\nabla \wedge A) \bullet ds + \iint_D (\nabla \wedge A) \wedge ds = 2\pi i\, NA(\rho_0) \quad \textbf{(22)}$$

or
$$\oint_l A \bullet dl = \iint_D (\nabla \wedge A) \bullet ds = 0 \quad \textbf{(23)}$$

$$\oint_l A \wedge dl = \iint_D (\nabla \bullet A) ds = 2\pi i\, NA(\rho_0) \quad \textbf{(24)}$$

$$\iint_D (\nabla \wedge A) \wedge ds = 0 \quad \textbf{(25)}$$

Formulas (18) and (22) unite Stokes' theorem with Cauchy's generalized integral formula in a four-dimensional pseudo-Euclidean space.

The physical meaning of the formulas (18) - (25) will be considered in the next section.

Physical interpretations

By comparing the formula (2.1) with (2.3) of Appendix 1, also taking into account the components of the electromagnetic field tensor F, we can write the formula (23) in the form:

$$\oint_l (A_0 dt - A_1 dx - A_2 dy - A_3 dz) = \iint_D ((E_x dx + E_y dy + E_z dz)dt - B_x dydz - B_y dxdz - B_z dxdy) = 0$$

or

$$\oint_l (A_0 dt - \boldsymbol{A} \bullet d\boldsymbol{r}) = \iint_D \boldsymbol{E} \bullet d\boldsymbol{r} dt - \iint_D \boldsymbol{B} \bullet \boldsymbol{n} ds = 0 \quad \textbf{(26)}$$

Simply put, the circulation of the 4x potential of the electromagnetic field $A=\gamma^i A_i(x_j)$ and the flux of the electric field ***E*** and the magnetic field ***B*** (more precisely, their difference) through domain D is equal to zero.

Equation (24) interests us better. We expand this equation into elements of the basis γ^i:

$$\oint_l A \wedge dl = i \oint_l (\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\sigma_1 (A_0 dx - A_1 dt) + \sigma_2 (A_0 dy - A_2 dt) + \sigma_3 (A_0 dz - A_3 dt)) +$$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\sigma_1 (A_3 dy - A_2 dz) + \sigma_2 (A_1 dz - A_3 dx) + \sigma_3 (A_2 dx - A_1 dy))) \quad \textbf{(27)}$$

$$\iint_D (\nabla \bullet A) ds = i \iint_D (\nabla \bullet A) (\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\sigma_1 dx + \sigma_2 dy + \sigma_3 dz) dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\sigma_1 dzdy + \sigma_2 dxdz + \sigma_3 dydx)) \quad \textbf{(28)}$$

$$2\pi i\, NA(\rho_0) = -2\pi A(\rho_0)(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\boldsymbol{n}_T + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{n}_r) \qquad \textbf{(29)}$$

Comparing (27), (28), and (29), simplifying and dividing by the elements of the matrix γ^i, we get:

$$\oint_{lT}(A_0 d\boldsymbol{r} - \boldsymbol{A}dt) = \iint_{DT}(\nabla \bullet A)\boldsymbol{n}_T ds_T = 2\pi i A(\rho_0)\,\boldsymbol{n}_T \qquad \textbf{(30)}$$

$\boldsymbol{n}_T = \sigma_1 \cos\alpha_{01} + \sigma_2 \cos\alpha_{02} + \sigma_3 \cos\alpha_{03}$ is the "time" vector of the surface normal.

$$\oint_{lXYZ}\mathbf{A} \times d\boldsymbol{r} = \iint_{DXYZ}(\nabla \bullet A)\boldsymbol{n}_r ds_r = 2\pi i A(\rho_0)\boldsymbol{n}_r \qquad \textbf{(31)}$$

$\boldsymbol{n}_r = \sigma_1 \cos\alpha_{32} + \sigma_2 \cos\alpha_{13} + \sigma_3 \cos\alpha_{21}$ is the "spatial" vector of the surface normal;

$\boldsymbol{A} \times d\boldsymbol{r}$ is an ordinary vector product of vectors.

$$\boldsymbol{n}_r ds_r = \sigma_1 dzdy + \sigma_2 dxdz + \sigma_3 dydz;\ \boldsymbol{n}_T ds_T = (\sigma_1 dx + \sigma_2 dy + \sigma_3 dz)dt; \quad lT \notin XYZ, DT \notin XYZ$$

Corollary:

Even if the 4th potential of the vector field $A=\gamma^i A_i(x_j)$ has a singularity at point ρ, and at other points of the domain D is analytic, then

1. According to the formula (23), the circulation $\oint_l A \bullet dl$ of the vector function $A=\gamma^i A_i(x_j)$ and the flow $\iint_D F \bullet ds$ of the electromagnetic tensor through the plane D are equal to zero. In particular, if $A_0 = 0$ and $\boldsymbol{A}$ are not time dependent, the form (23) is as follows:

$$\oint_{lXYZ} \boldsymbol{A} \bullet \boldsymbol{dr} = \iint_{DXYZ} \boldsymbol{B} \bullet \boldsymbol{n}_r ds = 0,$$

I.e., the circulation of the 3-dimensional potential of the electromagnetic field ***A*** and the flux of the magnetic field ***B*** are equal to zero. This is Gauss's theorem.

2. According to the formulas (30) and (31), the divergences through the surfaces *DT* and *XYZ* are equal to the 3-dimensional residues $(2\pi i A(\rho_0)\,\boldsymbol{n}_T, 2\pi i A(\rho_0)\boldsymbol{n}_r)$ of the $A=\gamma^i A_i(x_j)$ function. In particular, if $\boldsymbol{n}_r = const$, then we get the classic formula:

$$\iint_{DXYZ}(\nabla \bullet A)ds_r = 2\pi i A(\rho_0).$$

From a physical aspect, the divergence is the source of the field. So we can conclude that the multidimensional residues of the functionality found in the article means the source, i.e. current (charge) of the electromagnetic field.

Discussions and Conclusions

1. The relationship between the line and surface integrals in 3-dimensional Euclidean space and in Minkowski space was considered in Clifford algebra.

2. The Stokes theorem and the integral theorem and the Cauchy formula for hypercomplex numbers were generalized and combined.

3. The physical interpretation of the formulas (26) - (31) indicates a correspondence between the laws of electromagnetism and the theory of generalized hypercomplex analysis: between the circulation of the vector potential *A* and the flow of the electromagnetic tensor *F* with the generalized Stokes formulas and the Cauchy integral formula.

4. Formulas (30) - (31) establish the correspondence between the generalized 4-dimensional residues and the 4-dimensional electromagnetic current (charge). In other words, the electromagnetic current (charge) is a 3-dimensional residue of the vector function *A*.

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Appendix 1.

Taking into account formulas (1) and (2), we write the integrals $\oint_l \boldsymbol{A}\, d\boldsymbol{l}$ and $\iint_D (\boldsymbol{\nabla A})\mathrm{d}\boldsymbol{s}$ of equation (3) in the coordinate form:

$$\oint_l \boldsymbol{A} \bullet d\boldsymbol{l} = \sigma_0 \oint_l (A_1\, dx + A_2 dy + A_3 dz) \tag{1.1}$$

$$\oint_l \boldsymbol{A} \wedge d\boldsymbol{l} = i \oint_l (\sigma_1(A_2 dz - A_3 dy) + \sigma_2(A_3 dx - A_1 dz) + \sigma_3(A_1 dy - A_2 dx)) \tag{1.2}$$

$$\iint_D (\nabla\wedge \boldsymbol{A})\bullet \mathrm{d}\boldsymbol{s} = \sigma_0 \iint_D ((\frac{\partial A_2}{\partial z}-\frac{\partial A_3}{\partial y})dydz + (\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z})dzdx + (\frac{\partial A_1}{\partial y}-\frac{\partial A_2}{\partial x})dxdy) \quad (1.3)$$

$$\iint_D (\nabla\bullet \boldsymbol{A})\mathrm{d}\boldsymbol{s} = i \iint_D ((\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z})(\sigma_1 dydz + \sigma_2 dzdx + \sigma_3 dxdy)) \quad (1.4)$$

$$\iint_D (\nabla\wedge \boldsymbol{A})\wedge \mathrm{d}\boldsymbol{s} = \iint_D (\sigma_1(\frac{\partial A_2}{\partial z}-\frac{\partial A_3}{\partial y}) + \sigma_2(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}) + \sigma_3(\frac{\partial A_1}{\partial y}-\frac{\partial A_2}{\partial x}))\wedge(\sigma_1 dydz + \sigma_2 dzdx + \sigma_3 dxdy) = 0 \quad (1.5)$$

We equate equation (1.1) to (1.3) because both equations are "scalar" (σ_0). We also equate equation (1.2) to (1.4), since both of them are "vectors" (σ_α). Equation (1.5) is equal to zero. Now we will prove all this.

1. We project equation (4) (respective integrals) on the *X0Y*, *Y0Z*, *Z0X* planes (Fig. 1) and apply the proof of Green's theorem for each plane [**3**]. For example,

$$\oint_{l3}(A_1\,dx + A_2 dy) = \iint_{D3}(\frac{\partial A_1}{\partial y}-\frac{\partial A_2}{\partial x})dxdy) \text{ etc.}$$

2. We project the double integral on the right side of equation (5) onto the planes *X0Y*, *Y0Z*, and *Z0X* and transform it into the line integral. For example, (Fig.2)

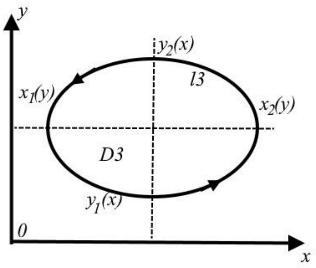

Fig.2.

$$i\iint_{D3}(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y})\sigma_3 dxdy = i\sigma_3\left(\iint_{D3}\frac{\partial A_1}{\partial x}dxdy + \iint_{D3}\frac{\partial A_2}{\partial y}dxdy\right) =$$

$$= i\sigma_3\int dy\,(A_1(x_2,y)-A_1(x_1,y)) + i\sigma_3\int dx\,(A_2(x,y_1)-A_2(x,y_2)) =$$

$$= i\oint_{l3}\sigma_3\,A_1 dy - i\oint_{l3}\sigma_3\,A_2\,dx = i\oint_{l3}\sigma_3\,(A_1 dy - A_2 dx) \qquad \text{etc.}$$

3. For equation (6) (or (1.5)) we also project the integrals onto the planes. For example,

$$\iint_{D3}(\nabla\wedge \boldsymbol{A})\wedge \mathrm{d}\boldsymbol{s} = \iint_{D3}(\sigma_1\sigma_2(\frac{\partial A_2}{\partial z}-\frac{\partial A_3}{\partial y})dzdx + \sigma_2\sigma_1(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z})dydz) = 0$$

Since $dz=0$ on the *X0Y* plane, the integral is also equal to zero. In addition, the thrice-outer product of the basis vectors in three-dimensional space is equal to zero: $\sigma_\alpha\wedge\sigma_\beta\wedge\sigma_\lambda\wedge\sigma_\mu = 0$, because $\alpha\neq\beta\neq\lambda\neq\mu$, and $\alpha, \beta, \lambda, \mu$ = 1,2,3.

Theorem 1 is proved.

Appendix 2

Proof of formula (13) (or (14)).

We decompose the *TXYZ* space, respectively, the $\oint_l Adl$ and $\iint_D (\nabla A)dS$ integrals into subspaces (planes) *X0Y*, *Y0Z*, *Z0X*, *T0X*, *T0Y*, *T0Z*.

1. We can write the left side of the equation (14) at the coordinates, separating it into the symmetric ($\oint_l A \bullet dl$) and antisymmetric ($\oint_l A \wedge dl$) parts.

$$\oint_l A \bullet dl = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix} \oint_l (A_0 dt - A_1 dx - A_2 dy - A_3 dz) \tag{2.1}$$

$$\oint_l A \wedge dl = \oint_l (\begin{bmatrix} 0 & \iota\sigma_1 \\ -\iota\sigma_1 & 0 \end{bmatrix} (A_0 dx - A_1 dt) + \begin{bmatrix} 0 & \iota\sigma_2 \\ -\iota\sigma_2 & 0 \end{bmatrix} (A_0 dy - A_2 dt) + + \begin{bmatrix} 0 & \iota\sigma_3 \\ -\iota\sigma_3 & 0 \end{bmatrix} (A_0 dz - A_3 dt) + \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix} (A_2 dx - A_1 dy) + \begin{bmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{bmatrix} (A_1 dz - A_3 dx) + + \begin{bmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{bmatrix} (A_2 dz - A_3 dy)) \tag{2.2}$$

2. Taking into account $\nabla \wedge A = F$ and separating in the same way the right side of the equation (14), we get:

$$\iint_D F \bullet dS = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix} \iint_D (F_{01} dtdx + F_{02} dtdy + F_{03} dtdz + F_{12} dxdy + F_{31} dxdz + F_{23} dydz) \tag{2.3}$$

$$\iint_D (\nabla \bullet A) dS = \iint_D (\nabla \bullet A)(\begin{bmatrix} 0 & \iota\sigma_1 \\ -\iota\sigma_1 & 0 \end{bmatrix} dtdx + \begin{bmatrix} 0 & \iota\sigma_2 \\ -\iota\sigma_2 & 0 \end{bmatrix} dtdy + \begin{bmatrix} 0 & \iota\sigma_3 \\ -\iota\sigma_3 & 0 \end{bmatrix} dtdz + + \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix} dydx + \begin{bmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{bmatrix} dxdz + \begin{bmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{bmatrix} dydz) \tag{2.4}$$

$$\iint_D F \wedge dS = \begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix} \iint_D (F_{01} dydz + F_{02} dzdx + F_{03} dxdy + F_{12} dtdz + F_{31} dtdy + F_{23} dtdx) \tag{2.5}$$

Other terms (integral (2.5)) are equal to zero.

Comparing (2.1) with (2.3), and also (2.2) with (2.4), we get the equations (14.1), (14.2), and (14.3).

It is now sufficient for us to consider all the subspaces and generalize the preceding theorems (case d=3) and apply the results of **Appendix 1**.

To prove equation (14.1), the integrals $\oint_l A \bullet dl$ and $\iint_D F \bullet dS$ must be expanded into integrals over the planes *X0Y*, *Y0Z*, *Z0X*, *T0X*, *T0Y*, *T0Z*, and the proofs of Green's theorem must be applied to each.

1. For example, for planes *X0Y*, *Y0Z*, *Z0X* there will be

$$\oint_{lX0Y} (-A_1 dx - A_2 dy) = \iint_{DX0Y} (\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x}) dxdy, \text{ etc.}$$

For the planes *T0X, T0Y, T0Z*, the proofs are a bit different, since these hyperplanes are complex, unlike *X0Y, Y0Z, Z0X*.

$$\oint_l (A_0 dt - A_1 dx) = \iint_D (\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t}) dt dx$$

Changing $A_1 \to iA_1, x \to ix$, etc., we get the usual Green's theorem:

$$\oint_l (A_0 dt + A_1 dx) = \iint_D (i\frac{\partial A_0}{\partial x} - i\frac{\partial A_1}{\partial t}) i dt dx = \iint_D (\frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial x}) dt dx, \text{ etc.}$$

Summing up all the integrals in all subspaces, we get the equation (14.1).

2. We now consider equation (14.2). Also, as in the previous case, we divide the integrals $_l\oint A \wedge dl$ and $_D\iint (\nabla \bullet A) dS$ into subspaces. We apply paragraph 2 of **Appendix 1**.

For example,

$$\begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix} \oint_{lX0Y} (A_2 dx - A_1 dy) = \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix} \iint_{DX0Y} (-\frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial y}) dy dx = -\begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix} (\iint_{DX0Y} \frac{\partial A_1}{\partial x} dy dx + \iint_{DX0Y} \frac{\partial A_2}{\partial y} dy dx),$$

etc.

Also for “time” planes:

$$\begin{bmatrix} 0 & \iota\sigma_3 \\ -\iota\sigma_3 & 0 \end{bmatrix} \oint_{lT0Z} (A_0 dz - A_3 dt) = \begin{bmatrix} 0 & \iota\sigma_3 \\ -\iota\sigma_3 & 0 \end{bmatrix} \iint_{DT0Z} (\frac{\partial A_0}{\partial t} - \frac{\partial A_3}{\partial z}) dt dz = \begin{bmatrix} 0 & \iota\sigma_3 \\ -\iota\sigma_3 & 0 \end{bmatrix} (\iint_{DT0Z} \frac{\partial A_0}{\partial t} dt dz - \iint_{DT0Z} \frac{\partial A_3}{\partial z} dt dz), \text{ etc.}$$

3. For equation (14.3), we also apply point 3 from **Appendix 1**. Formula (14.3) is also proved by dividing into purely "spatial" and "time" hyperplanes.

Theorem 4, i.e. equation (14), respectively (14.1), (14.2) and (14.3) are proved.

Note This paper is a partial translation and a more mathematically general and strictly proven version of the author’s article in Russian, published in the reviewed journal: https://sci-article.ru/stat.php?i=1562874175