The alternative formalism based on the generalized Clifford algebra

Babaev A. Kh.

Abstract: The article presents the unification of two Maxwell's systems equations (homogeneous and inhomogeneous) within the generalized Clifford algebra. In this new formalism, an electromagnetic current and certain gauges acquire a geometric meaning associated with the properties of space.

Keywords: Clifford algebra; outer and inner products of vectors; Maxwell's equations; 4-dimensional electromagnetic current; Lorentz gauge; gauge invariance.

Introduction

It is known that in the covariant form, Maxwell's equations consist of two independent systems [1]:

• homogeneous – $E^{ijkl}F_{ij;k} = 0$;

• inhomogeneous –$F^{ik}_{;k} = J^i$.

here F_{ij} is an electromagnetic field tensor; J_i is a 4-dimensional electromagnetic current density; $E^{ijkl} = \frac{\varepsilon^{ijkl}}{\sqrt{-g}}$ $(\varepsilon^{0123} = +1)$ is the contravariant antisymmetric tensor of the fourth rank or Levi-Civita symbol; (-g) is a determinant of the metric tensor; ε^{ijkl} is a antisymmetric tensor of the fourth rank in the orthonormal basis; $F_{ij;k} = \mathcal{D}_k F_{ij} = \frac{\mathcal{D}F_{ij}}{\partial q^k}$ is a covariant derivative of the electromagnetic field tensor by argument (coordinate) q^k.

Relevance.

It would make sense to combine two independent systems of Maxwell's equations into one equation. Marcel Ries [2] for the first time combined these systems in Minkowski space. But he did not connect the four-current with the properties and special points of space, i.e., he "threw out" the features of space: curvature and special points. This very limited the ability of Clifford algebra to combine the systems of Maxwell's equations and 4-current.

In classical physics, electric charge and current are not related to the explicit properties of space, such as the space's metric (curvature) and special points.

The scientific novelty of this research lies in the fact that two independent systems of Maxwell's equations are combined into a single one for any space. Also, the 4-current and basic calibration conditions (Lorentz, Coulomb calibration, etc.) are associated with special points in space where the field potential is not defined and/or there is no limit.

Theoretical foundations

As a measure of the change in the vector field, we introduce the concept of local inhomogeneity (B) of the vector field with potential A:

$$B = \nabla A \tag{1}$$

here $\nabla \equiv e^i D/\partial q^i$ is a Del or nabla operator; $A = e^i A_i$ is the expansion of the vector-potential *A* in the vectors e^j of the curvilinear basis.

Latin letters take values from 0 to 3: *i,j* ... = 0,1,2,3., Greek - from 1 to 3: *α,β* ...= 1,2,3.

As an orthonormal basis, we take the canonical basis:

$$\gamma_i\gamma_j + \gamma_j\gamma_i = \pm I\delta_{ij} \tag{2}$$

here γ_i is the Dirac matrices; δ_{ij} is Kronecker delta; *I* is a 4x4 unit matrix.

According to the signature of space, in particular Minkowski, if *i*=0 (or *j*=0), we take the sign "+" in equality (2), if not, we take the sign "-".

Instead of the usual scalar and vector products, we use Clifford products [3] in the products of basis vectors $e_j = \gamma_k \partial_j X^k$:

$$e_i e_j = e_i \bullet e_j + e_i \wedge e_j \tag{3}$$

here

inner product – $$e_i \bullet e_j = 0.5(e_i e_j + e_j e_i) \tag{4}$$

outer product – $$e_i \wedge e_j = 0.5(e_i e_j - e_j e_i) \tag{5}$$

Remark. The inner product of vectors is not a scalar product, as is often mistakenly used in the scientific literature. The inner product of the two vectors is a second-rank symmetric tensor, and the scalar product of the same vectors is simply a scalar, i.e., a trace of the second-rank symmetric tensor.

X^k are functions of $\{q^i\}$, i.e., functions of transition from the curvilinear to the orthonormal coordinate system.

Taking into account (4) and (5), the inhomogeneity of the vector field (1) in the coordinate form has the form:

$$B = e^i \bullet e^j \mathcal{D}_i A_j + e^i \wedge e^j \mathcal{D}_i A_j \tag{6}$$

here $e^i \bullet e^j = g^{ij}$ is a metric tensor; $e^i \wedge e^j$ is an antisymmetric second-rank tensor or bivector.

In the general case

a) The antisymmetric tensor of rank $e_i \wedge e_j \wedge e_k \wedge e_n$ is dual to the pseudoscalar in the 4-dimensional space:

$$e_i \wedge e_j \wedge e_k \wedge e_n = -\gamma E_{ijkn} \tag{7.1}$$

$$e^i \wedge e^j \wedge e^k \wedge e^n = \gamma E^{ijkn} \tag{7.2}$$

here $\gamma = \gamma_0\gamma_1\gamma_2\gamma_3$; $E_{ijkn} = \sqrt{-g}\,\varepsilon_{ijkn}, (\varepsilon_{0123} = -1)$; $E^{ijkn} = \frac{\varepsilon^{ijkn}}{\sqrt{-g}}, (\varepsilon^{0123} = +1)$ is absolutely antisymmetric tensors (or Levi-Civita symbol) of the 4th rank in covariant and contravariant form;

The product $e_0 \wedge e_1 \wedge e_2 \wedge e_3$ is equal to the "4-dimensional volume" built from the vectors e_0, e_1, e_2, e_3.

Proof of equalities (7.1) and (7.2) is given in Appendix 1.

b) In 4-dimensional space, the product $e_i \wedge e_j \wedge e_k$ (3rd rank antisymmetric tensor) is dual to the pseudovector:

$$e_i \wedge e_j \wedge e_k = -\gamma E_{ijkn} e^n \qquad (8.1)$$

$$e^i \wedge e^j \wedge e^k = \gamma E^{ijkn} e_n \qquad (8.2)$$

Proof of equalities (8.1) and (8.2) is given in Appendix 2.

c) In 4-dimensional space, the product $e_i \wedge e_j$ (2nd rank antisymmetric tensor) is dual to itself (2nd rank antisymmetric pseudotensor):

$$e_i \wedge e_j = -\gamma E_{ijkn} e^k \wedge e^n \qquad (9.1)$$

$$e^i \wedge e^j = \gamma E^{ijkn} e_k \wedge e_n \qquad (9.2)$$

The formulas (9.1) and (9.2) can be proved the same way as the previous cases, so we will not bother the reader with calculations.

Results

Maxwell 's equations.

To get the unified equation of electromagnetism, we take the gradient of the equation (1):

$$\nabla B = \nabla(\nabla A) \qquad (10)$$

According to Clifford's product, we have

$$\nabla B = \nabla(\nabla \bullet A + \nabla \wedge A) = \nabla(\nabla \bullet A) + \nabla \bullet (\nabla \wedge A) + \nabla \wedge \nabla \wedge A$$

or

$$\nabla B = \nabla(\nabla \bullet A) + \nabla \bullet (\nabla \wedge A) + \nabla \wedge \nabla \wedge A \qquad (11)$$

Equation (11) is the unified equation of electromagnetism.

Homogeneous system of Maxwell 's equations.

Theorem:

– the following statement is true

$$\nabla \wedge \nabla \wedge A = 0 \qquad (12)$$

and the equation (12) is equivalent to the homogeneous Maxwell equation.

If we take into account that

$$\mathrm{F} = \nabla \wedge A = e^i \wedge e^j (\mathcal{D}_i A_j - \mathcal{D}_j A_i) \qquad (13),$$

then from (12) we obtain the classical form of the homogeneous Maxwell equation:

$$\nabla \wedge F = 0 \qquad (14)$$

Equation (14) is a homogeneous Maxwell equation.

The proof of statement (14) is given in Appendix 3.

4-dimensional electromagnetic current

Taking into account (12), we write equation (11) in the form:

$$\nabla B = \nabla(\nabla \bullet A) + \nabla \bullet (\nabla \wedge A) \tag{15}$$

We denote the 4th current as

$$J = \nabla(\nabla \bullet A) \tag{16}$$

According to (16), the 4th current has a clear geometric meaning:

The 4th current is the 4th gradient from the 4th divergence of the potential field. The 4th current exists only when the divergence is not constant; i.e., $\nabla \bullet A \neq const$ is the main condition of current existence. The space can have so-called "holes" and/or "clumps" (inhomogeneities) if only $\nabla \bullet A \neq const$. Let's call these "holes" "outflows" and/or "inflows". So the electric charge is either "outflows" is a positive charge (+), or "inflows" is a negative one (-).

Lorenz gauge

$\nabla \bullet A = const(= 0)$ means there is no 4-current in the research field. In particular, the Coulomb gauge $\boldsymbol{\nabla} \bullet \boldsymbol{A} = const(= 0)$ also means no three-dimensional current in the magnetostatic problems, where the time component A_0 is ignored or assumed to be zero.

Gauge invariance

If we add the 4th gradient of the scalar function u to the potential A

$$A' = A + \nabla u \tag{17}$$

and require the condition

$$\nabla \bullet \nabla u = 0 \qquad (18),$$

then the unified electromagnetism equation (10) will be invariant.

The transformations (17) are gauge invariance.

Inhomogeneous system of Maxwell equations

Taking into account the new definition of 4th currents (16) and the electromagnetic field tensor (13), we write the unified equation of electromagnetism (15) in the form:

$$\nabla B = J + \nabla \bullet F \tag{19}$$

Since the 3rd rank tensor ∇B in 4-dimensional space is dual to the pseudovector, we can write as

$$\nabla B = \mu T \bullet A \tag{20}$$

here T is energy–momentum tensor; μ is constant factor.

The inner product $T \bullet A$ is a dual vector to the ∇B.

Then from equation (19) we will get the inhomogeneous Maxwell equation:

$$\nabla \bullet F = \mu T \bullet A - J \tag{21}$$

If $\mu T \bullet A \cong 0$, then we will get the classical expression for the inhomogeneous Maxwell equation:

$$\nabla \bullet F = -J \quad (22)$$

If $\mu T \bullet A \neq 0$, then the classical Maxwell law (22) does not hold. More correctly, it takes more of the universal form. According to (21), at high energies, the energy-momentum tensor contributes to the 4th current.

Denoting the right side of the equation (21) as

$$Q = \mu T \bullet A - J \quad (23)$$

we get from (21)

$$\nabla \bullet F = Q \quad (24)$$

Here Q is the effective 4th current (in particular, the effective charge [4]). According to (23), the effective current (charge) depends on the energy-momentum and is not a constant value.

Thus, we have shown that two independent Maxwell systems (12) and (22) are parts of a single equation (10) or (11).

Discussions and Conclusions

1. Two systems of Maxwell's equations are unified in one single equation (11). In other words, the homogeneous (14) and inhomogeneous (24) systems of the Maxwell equations are the parts of a single equation (11);
2. 4-x current $J = \nabla(\nabla \bullet A)$ (16) has an explicit geometric meaning: 4-x current J is the gradient from the divergence of the potential A, i.e., includes singularities of space. The change in time of "outflow" and/or "inflow" in space is a positive and/or negative electric charge.
3. The imposition of restrictions, such as Lorentz, Coulomb gauges on equations is nothing more than ignoring the singularity, i.e., current. The condition (18) in gauge invariance (17) means that oscillations and/or waves do not affect the four- current.
4. A new form of the inhomogeneous Maxwell system (21) means that energy-momentum contributes to the 4- current. At low energies, $\mu T \bullet A \approx 0$ (classical case), and can be $\mu T \bullet A - J > 0$ at high energies. Probably, at high energies, the "running" of the Weinberg angle, predicted in the Standard Model [5] and confirmed in experiments [6], and in general, the "running of constants" is maybe related to this contribution. It is possible that the confinement [7] in the theory of quarks is explained by the same contribution. For large contributions to the 4-current, the difference ($\mu T \bullet A - J$) will be so great that the appearance of free single quarks will become impossible.

References:

1. Landau L.D., Lifshitz E. M., Course of Theoretical Phisics, Third English edition 1971, Vol.2, pages 255 - 257, Library of Congress Catalog Card No. 73-140427
2. Riesz, Marcel (1993) (1958), Clifford numbers and spinors, Fundamental Theories of Physics, 54, Kluwer Academic Publishers Group, ISBN 978-0-7923-2299-3, MR 1247961.

3. Chris J. L. Doran. Geometric Algebra and its Application to Mathematical Physics. Sidney Sussex College. A dissertation submitted for the degree of Doctor of Philosophy in the University of Cambridge. February 1994, pages 4-6.

4. Clementi, E.; Raimondi, D. L. (1963). "Atomic Screening Constants from SCF Functions". *J. Chem. Phys*. 38 (11): 2686–2689. Bibcode:1963JChPh..38.2686C. doi:10.1063/1.1733573

5. G. P. Zeller *et al.* Phys. Rev. Lett. 88, 091802 – Published 12 February 2002; Erratum Phys. Rev. Lett. 90, 239902 (2003)

6. P. L. Anthony et al, SLAC E158 Coll., "Precision measurement of the weak mixing angle in Moller scattering", e-print hep--ex/0504049.

7. R. Alkofer und J. Greensite: Quark Confinement: The Hard Problem of Hadron Physics. In: Journal of Physics. G, Nr. 34, 2007, doi:10.1088/0954-3899/34/7/S02, arxiv:hep-ph/0610365

Appendix 1

Proof of formulas (7.1) and (7.2):

We expand each vector e_i of the generalized basis in terms of the canonical basis and write the product $e_i \wedge e_j \wedge e_k \wedge e_n$ in the form:

$$e_i \wedge e_j \wedge e_k \wedge e_n = (\partial_i X^p \partial_j X^q \partial_k X^r \partial_n X^s)\gamma_p \gamma_q \gamma_r \gamma_s$$

Here *i, j, k, n* (also *p, q, r, s*) are not equal to each other and take the values 0;1;2;3.

Simple calculations show that

$$e_i \wedge e_j \wedge e_k \wedge e_n = \begin{vmatrix} \partial_i X^0 & \partial_i X^1 & \partial_i X^2 & \partial_i X^3 \\ \partial_j X^0 & \partial_j X^1 & \partial_j X^2 & \partial_j X^3 \\ \partial_k X^0 & \partial_k X^1 & \partial_k X^2 & \partial_k X^3 \\ \partial_n X^0 & \partial_n X^1 & \partial_n X^2 & \partial_n X^3 \end{vmatrix} \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{vmatrix} \partial_i X^0 & \partial_i X^1 & \partial_i X^2 & \partial_i X^3 \\ \partial_j X^0 & \partial_j X^1 & \partial_j X^2 & \partial_j X^3 \\ \partial_k X^0 & \partial_k X^1 & \partial_k X^2 & \partial_k X^3 \\ \partial_n X^0 & \partial_n X^1 & \partial_n X^2 & \partial_n X^3 \end{vmatrix} \gamma$$

We square this determinant:

$$\begin{vmatrix} \partial_i X^0 & \partial_i X^1 & \partial_i X^2 & \partial_i X^3 \\ \partial_j X^0 & \partial_j X^1 & \partial_j X^2 & \partial_j X^3 \\ \partial_k X^0 & \partial_k X^1 & \partial_k X^2 & \partial_k X^3 \\ \partial_n X^0 & \partial_n X^1 & \partial_n X^2 & \partial_n X^3 \end{vmatrix}^2 = \begin{vmatrix} \partial_i X^0 & \partial_i X^1 & \partial_i X^2 & \partial_i X^3 \\ \partial_j X^0 & \partial_j X^1 & \partial_j X^2 & \partial_j X^3 \\ \partial_k X^0 & \partial_k X^1 & \partial_k X^2 & \partial_k X^3 \\ \partial_n X^0 & \partial_n X^1 & \partial_n X^2 & \partial_n X^3 \end{vmatrix} \begin{vmatrix} \partial_i X^0 & \partial_i X^1 & \partial_i X^2 & \partial_i X^3 \\ \partial_j X^0 & \partial_j X^1 & \partial_j X^2 & \partial_j X^3 \\ \partial_k X^0 & \partial_k X^1 & \partial_k X^2 & \partial_k X^3 \\ \partial_n X^0 & \partial_n X^1 & \partial_n X^2 & \partial_n X^3 \end{vmatrix}^T$$

Simplifying and extracting the last expression from the root and generalizing, we obtain formula (7.1).

$$\begin{vmatrix} \partial_i X^0 & \partial_i X^1 & \partial_i X^2 & \partial_i X^3 \\ \partial_j X^0 & \partial_j X^1 & \partial_j X^2 & \partial_j X^3 \\ \partial_k X^0 & \partial_k X^1 & \partial_k X^2 & \partial_k X^3 \\ \partial_n X^0 & \partial_n X^1 & \partial_n X^2 & \partial_n X^3 \end{vmatrix}^2 = -\begin{bmatrix} g_{ii} & g_{ij} & g_{ik} & g_{in} \\ g_{ji} & g_{jj} & g_{jk} & g_{jn} \\ g_{ki} & g_{kj} & g_{kk} & g_{kn} \\ g_{ni} & g_{nj} & g_{nk} & g_{nn} \end{bmatrix} = -g$$

The proof of formula (7.2) is similar.

We have proved assertions (7.1) and (7.2).

Appendix 2

Proof of formulas (8.1) and (8.2):

It is known that in 4-dimensional space the antisymmetric tensor of the 3rd rank is dual to the pseudovector. We write this in the form:

$$e_i \wedge e_j \wedge e_k = \Omega e_m \quad (2A)$$

Ω is a yet unknown coefficient.

We multiply both sides of equality (2A) by e_n on the right:

$$e_i \wedge e_j \wedge e_k \wedge e_n = \Omega e_m \bullet e_n$$

Simplify: $-\gamma E_{ijkl} = \Omega g_{mn}$

Then: $\Omega = -\gamma g^{mn} E_{ijkl}$

Substituting this expression for Ω in equality (2A), we obtain equality (8.1). The proof of equality (8.2) is similar.

We have proved assertions (8.1) and (8.2).

Appendix 3

Proof of formulas (12) or (14):

1. Proof of assertion (12), i.e., $\nabla \wedge \nabla \wedge A = 0$.

Taking into account (8.2), we can write formula (12) in the coordinate form:

$$E^{kijn} \mathcal{D}_k \mathcal{D}_i A_j = 0$$

Changing the places of the indices *i*,*j*,*k* in this equation and simplifying, we get:

$$E^{kijn}(\mathcal{D}_k \mathcal{D}_i A_j + \mathcal{D}_j \mathcal{D}_k A_i + \mathcal{D}_i \mathcal{D}_j A_k - \mathcal{D}_i \mathcal{D}_k A_j - \mathcal{D}_j \mathcal{D}_i A_k - \mathcal{D}_k \mathcal{D}_j A_i) = 0$$

It is known that $\mathcal{D}_i \mathcal{D}_j A_k - \mathcal{D}_j \mathcal{D}_i A_k = -R^p_{kij} A_p$; R^p_{kij} is the Riemann tensor.

Substituting the Riemann tensor expression into the previous equation, we get

$$-R^p_{kij} A_p - R^p_{jki} A_p - R^p_{ijk} A_p = -A_p \left(R^p_{kij} + R^p_{jki} + R^p_{ijk}\right) = 0$$

This equation is indeed equal to zero, since the expression in brackets is identically equal to zero.

2. Proof of equivalence (12) and the homogeneous Maxwell equation.

We transform the formula $E^{kijn} \mathcal{D}_k \mathcal{D}_i A_j = 0$:

$$0 = E^{kijn} \mathcal{D}_k \mathcal{D}_i A_j = E^{kijn} \mathcal{D}_k \mathcal{D}_i A_j + E^{kijn} \mathcal{D}_k \mathcal{D}_i A_j = E^{kijn} \mathcal{D}_k \mathcal{D}_i A_j + E^{kjin} \mathcal{D}_k \mathcal{D}_j A_i =$$

$$= E^{kijn}\left(\mathcal{D}_k \mathcal{D}_i A_j - \mathcal{D}_k \mathcal{D}_j A_i\right) = E^{kijn} \mathcal{D}_k \left(\mathcal{D}_i A_j - \mathcal{D}_j A_i\right) E^{kijn} \mathcal{D}_k F_{ij} = 0$$

The statements are proven.

Note. This paper is an improved version and partial translation of an author's article in Russian, published in a peer-reviewed journal: SCI-ARTICLE (40, December, 2016).

https://sci-article.ru/stat.php?i=1480330789